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Topological Entropy and Topologically Mixing Property in b-Topological Spaces

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ABSTRACT

In this paper, we investigate the concepts of topological entropy (briefly top. ent.) and topologically mixing (briefly top. mix.) property in the context of b-topological spaces (briefly b- top. sps.). We introduce the notion of b-top. ent.as a measure of complexity for dynamical systems (briefly dy. sy.) defined on b-top. sps. We establish its properties and provide examples to illustrate its behavior. Furthermore, we define and study the top. mix. property in the setting of b-top. sps. and explore its connections with other dynamical properties. We present several results characterizing top. mix. and discuss its implications for the dynamics of b-topological systems.

Keywords: b-Topological Spaces, Topological Entropy, Topologically Mixing Property, Dynamical Systems

INTRODUCTION

Top. ent. and top. mix. property have been extensively studied in the field of dy.sy. and topology. These concepts provide insights into the complexity and behavior of cont. transformations on topological spaces. The study of top. ent. and top. mix. property has found applications in various areas, including chaos theory, data analysis, and cryptography.
In recent years, there has been growing interest in extending these concepts to b-top. sps. b-top. sps., introduced by Franklin and Rajagopalan, are a generalization of topological spaces that allow for a more flexible notion of convergence. This generalization opens up new possibilities for investigating the behavior of cont. maps on b-top. sps. and exploring their dynamical properties.
The main of this paper is to investigate the notions of top. ent. and top. mix. property in the context of b-topological spaces. We aim to provide a comprehensive understanding of these concepts and their implications in the realm of b-top. sy.

Preliminaries

Let $(\mathbb{U},\mathfrak{L})$ be a top. sp., and let $\mathcal{U}$ be a sub set of $\mathbb{U}$,we used the concept of closure and interior of sets (briefly C and I respectively) for definition as follows

Definition 2-1

1. Let $\mathcal{U} \subset \mathbb{U}$,we called $\mathcal{U}$ is semi-open set [1] if $\mathcal{U} \subset \mathrm{C}(\mathrm{I}(\mathcal{U}))$. $\mathcal{U}$ is semi-closed set if $I\,(C(\mathcal{U})) \subset \mathcal{U}$
2. Let $\mathcal{U} \subset \mathbb{U}$, we called $\mathcal{U}$ is pre-open set [2] if $\mathcal{U} \subset \mathrm{I}(C\,(\mathcal{U}))$. $\mathcal{U}$ is pre-closed set if $C(I\,(\mathcal{U})) \subset \mathcal{U}$

3. Let $\mathcal{U} \subset \mathbb{U}$, we called $\mathcal{U}$ is b-open set [3] if $\mathcal{U} \subset C\,(I(\mathcal{U})) \cup \mathrm{I}(C\,(\mathcal{U}))$, . $\mathcal{U}$ is b-closed set if $\mathrm{I}(\mathrm{C}(\mathcal{U})) \cap \mathrm{C}(\mathrm{I}\,(\mathcal{U})) \subset \mathcal{U}$,

New we given new definition of semi- (pre, b) -open set

Definitions 2-2

1. Let $\mathcal{U} \subset \mathbb{U}$, we called $\mathcal{U}$ is semi-open set if $\mathcal{U} \subset C(\mathcal{U})$.
2. Let $\mathcal{U} \subset \mathbb{U}$, we called $\mathcal{U}$ is pre-open set if $\mathcal{U} \subset I(\mathcal{U})$.
3. Let $\mathcal{U} \subset \mathbb{U}$, we called $\mathcal{U}$ is b-open set if $\mathcal{U} \subset C(\mathcal{U}) \cup I(\mathcal{U})$.

In the propositions follows we given the relation between the original definitions and our definitions.

Proposition 2-3 Let $(\mathbb{U}, \mathfrak{L})$ be a top. sp., and let $\mathcal{U}$ be a sub set of $\mathbb{U}$,then set $\mathcal{U}$ is called a semi-open set if and only if the following two conditions are equivalent:

1-$\mathcal{U} \subset C(I(\mathcal{U}))$,

2- $\mathcal{U} \subset C(\mathcal{U})$.

Proof: Direction 1: ($\mathcal{U}$ is a semi-open) $\Rightarrow$ ($\mathcal{U} \subset C(I(\mathcal{U}))$ and $\mathcal{U} \subset C(\mathcal{U})$)

Assume $\mathcal{U}$ semi-open set. We want to prove that $\mathcal{U}$ satisfies both conditions. Since $\mathcal{U}$ is semi-open, it means $\mathcal{U} \subset C(I(\mathcal{U}))$ (by definition). This satisfies the first condition.

To show that $\mathcal{U} \subset C(\mathcal{U})$, let's consider an arbitrary point $x \in \mathcal{U}$. We need to prove that x is in the closure of $\mathcal{U}$. Since $x \in \mathcal{U}$, x is either an interior point or a limit point of $\mathcal{U}$.

Case 1: x is an interior point of $\mathcal{U}$. there exists an open set U such that $x \in U \subset \mathcal{U}$. Since $U \subset \mathcal{U}$, U is contained in the closure of $\mathcal{U}$, i.e., $U \subset C(\mathcal{U})$. Therefore, $x \in C(\mathcal{U})$.

Case 2: x is a limit point of $\mathcal{U}$.

Since x is a limit point of $\mathcal{U}$, every open set containing x intersects $\mathcal{U}$ at a point other than x. This implies that for any open set V containing x, we have $V \cap \mathcal{U} \neq \emptyset$. In particular, $V \cap (\mathcal{U} \setminus \{x\}) \neq \emptyset$.

Consider the set $U = \mathrm{C}(\mathcal{U}\setminus\{x\})$. U is an open set because it is the complement of a closed set $(\mathcal{U}\setminus\{x\})$. Since $V \cap (\mathcal{U}\setminus\{x\}) \neq \emptyset$ for any open set V containing x, it means that V intersects $\mathcal{U}\setminus\{x\}$, which implies that V intersects U. Therefore, x is a limit point of U.We know that U is an open set containing x and x is a limit point of U. This means that $x \in C(U) = C(C(\mathcal{U}\setminus\{x\}))$. But $\mathrm{C}(\mathcal{U}\setminus\{x\})$ is a subset of $C(\mathcal{U})$ because removing an element does not affect the closure. Therefore, $x \in \mathrm{C}(\mathcal{U})$.In both cases, we have shown that $x \in \mathrm{C}(\mathcal{U})$, so $\mathcal{U} \subset \mathrm{C}(\mathcal{U})$, satisfying the second condition. Hence, if $\mathcal{U}$ is a semi-open set, it implies that $\mathcal{U} \subset \mathrm{C}(\mathrm{I}(\mathcal{U}))$ and $\mathcal{U} \subset \mathrm{C}(\mathcal{U})$.

Direction 2: ($\mathcal{U} \subset \mathrm{C}(\mathrm{I}(\mathcal{U}))$ and $\mathcal{U} \subset \mathrm{C}(\mathcal{U})$) $\Rightarrow$ ($\mathcal{U}$ is a semi-open set)

Assume $\mathcal{U} \subset \mathrm{C}(\mathrm{I}(\mathcal{U}))$ and $\mathcal{U} \subset \mathrm{C}(\mathcal{U})$). We want to prove that $\mathcal{U}$ is a semi-open set.

Let x be an arbitrary point in $\mathcal{U}$. We need to show that there exists an open set U such that $x \in U \subset \mathrm{C}\,(\mathcal{U})\setminus \mathcal{U}$.

Since $\mathcal{U} \subset C(I(\mathcal{U}))$, we have $\mathcal{U} \subset C(\mathcal{U}) \subset C(I(\mathcal{U}))$. This means that x is either an interior point or a limit point of $\mathrm{I}(\mathcal{U})$.

Case 1: x is an interior point of $\mathrm{I}(\mathcal{U})$.

In this case, there exists an open set V such that $x \in V \subset \mathrm{I}(\mathcal{U})$. Since $\mathrm{I}(\mathcal{U}) \subset \mathcal{U}$, we have $x \in V \subset \mathcal{U}$. Also, $V \subset \mathrm{C}(\mathcal{U})$ because $\mathrm{C}(\mathcal{U})$ contains $\mathrm{I}(\mathcal{U})$. Therefore, $U = V$ satisfies the condition $x \in U \subset \mathrm{C}(\mathcal{U}) \setminus \mathcal{U}$.

Case 2: x is a limit point of $\mathrm{I}\,(\mathcal{U})$.

Since x is a limit point, every open set containing x intersects $\mathrm{I}(\mathcal{U})$ at a point other than x. This means that for any open set V containing x, we have $V \cap \mathrm{I}(\mathcal{U}) \neq \emptyset$.

Consider the set $U = \mathrm{C}\ (\mathrm{I}(\mathcal{U}))\setminus \mathrm{I}(\mathcal{U})$. U is an open set because it is the complement of a closed set $(\mathrm{I}(\mathcal{U}))$. Since $V \cap \mathrm{I}(\mathcal{U}) \neq \emptyset$ for any open set V containing x, it means that V intersects $\mathrm{I}(\mathcal{U})$, which implies that V intersects U. Therefore, x is a limit point of U.
We know that U is an open set containing x, and x is a limit point of U. This means that $x \in \mathrm{C}(U) = \mathrm{C}(\mathrm{C}(\mathrm{I}(\mathcal{U})) \setminus \mathrm{I}(\mathcal{U}))$. But $\mathrm{C}(\mathrm{C}(\mathrm{I}(\mathcal{U})) \setminus \mathrm{I}(\mathcal{U}))$ is a subset of $\mathrm{C}(\mathcal{U})$ because removing an element does not affect the closure. Therefore, $x \in \mathrm{C}(\mathcal{U})$.
In both cases, we have shown the existence of an open set U such that $x \in U \subset \mathrm{C}(\mathcal{U}) \setminus \mathcal{U}$. Hence, $\mathcal{U}$ is a semi-open set.
Therefore, if $\mathcal{U} \subset \mathrm{C}(\mathrm{I}(\mathcal{U}))$ and $\mathcal{U} \subset \mathrm{C}(\mathcal{U})$, it implies that $\mathcal{U}$ is a semi-open set.

Examples 2-3:

1. Let $\mathcal{U} = (0,1)$ be a set in the real numbers. Here, $\mathrm{I}(\mathcal{U})$ =(0, 1), and $\mathrm{C}(\mathcal{U}) = [0,1]$. $\mathcal{U}$ satisfies the condition $\mathcal{U} \subset C(I(\mathcal{U}))$ since $\mathcal{U}$ is contained in its closure. Additionally, $\mathcal{U}$ also satisfies the condition $\mathcal{U} \subset C(\mathcal{U})$ since $\mathcal{U}$ is already closed. Therefore, $\mathcal{U}$ is a semi-open set.
2. Consider the set $\mathcal{U} = [0,1) \cup \{2\}$ in the real numbers. $\mathrm{I}(\mathcal{U}) = (0,1), \mathrm{C}(\mathcal{U}) = [0,1] \cup \{2\}$. $\mathcal{U}$ satisfies the condition $\mathcal{U} \subset \mathrm{C}(\mathrm{I}(\mathcal{U}))$ since $\mathcal{U}$ is contained in its closure. Additionally, $\mathcal{U}$ also satisfies the condition $\mathcal{U} \subset \mathrm{C}(\mathcal{U})$ since $\mathcal{U}$ is already closed. Therefore, $\mathcal{U}$ is a semi-open set.

Proposition 2-4:
A set $\mathcal{U}$ is called a pre-open set if and only if the following two conditions are equivalent:
1- $\mathcal{U} \subset I(C(\mathcal{U}))$, where C denotes the closure and I denotes the interior of $\mathcal{U}$.
2- $\mathcal{U} \subset I(\mathcal{U})$, where I denotes the interior of $\mathcal{U}$.

Direction 1: ($\mathcal{U}$ is a pre-open set) $\Rightarrow$ $(\mathcal{U} \subset I(C(\mathcal{U}))\ and\ \mathcal{U} \subset I(\mathcal{U}))$
Assume $\mathcal{U}$ is a pre-open set. We want to prove that $\mathcal{U}$ satisfies both conditions. Since $\mathcal{U}$ is pre-open, it means $\mathcal{U} \subset I(C(\mathcal{U}))$ (by definition). This satisfies the first condition.
To show that $\mathcal{U} \subset I(\mathcal{U})$, let's consider an arbitrary point $x \in \mathcal{U}$. We need to prove that x is in the interior of $\mathcal{U}$.Since $x \in \mathcal{U}$, x is an interior point of $\mathcal{U}$. This means that there exists an open set U such that $x \in U \subset \mathcal{U}$. Since U is an open set contained in $\mathcal{U}$, it is also contained in the interior of $\mathcal{U}$, i.e., $U \subset I(\mathcal{U})$. Therefore, $x \in I(\mathcal{U})$. Hence, if $\mathcal{U}$ is a pre-open set, it implies that $\mathcal{U} \subset I(C(\mathcal{U}))$ and $\mathcal{U} \subset I(\mathcal{U})$.
Direction 2: $(\mathcal{U} \subset I(C(\mathcal{U}))\ and\ \mathcal{U} \subset I(\mathcal{U}))$ $\Rightarrow$ ($\mathcal{U}$ is a pre-open set)
Assume $\mathcal{U} \subset I(C(\mathcal{U}))$ and $\mathcal{U} \subset I(\mathcal{U})$. We want to prove that $\mathcal{U}$ is a pre-open set.
Let x be an arbitrary point in $\mathcal{U}$. We need to show that there exists an open set U such that $x \in U \subset C(\mathcal{U})$.
Since $\mathcal{U} \subset I(C(\mathcal{U}))$, it means that $\mathcal{U}$ is contained in the interior of the closure of $\mathcal{U}$. Therefore, $x \in \mathcal{U}$ implies x is an interior point of the closure of $\mathcal{U}$, i.e., $x \in I(C(\mathcal{U}))$.Since $\mathcal{U} \subset I(\mathcal{U})$, it means that $\mathcal{U}$ is contained in its interior. Therefore, $x \in \mathcal{U}$ implies x is an interior point of $\mathcal{U}$, i.e., $x \in I(\mathcal{U})$.
Combining these two facts, we have $x \in I(C(\mathcal{U})) \cap I(\mathcal{U})$. Since the intersection of two open sets is an open set, we can define $U = I(C(\mathcal{U})) \cap I(\mathcal{U})$ as an open set containing x.Since U is an open set contained in both the closure of $\mathcal{U}$ and $\mathcal{U}$, it means that $U \subset C(\mathcal{U})$ and $U \subset \mathcal{U}$.
Therefore, we have shown the existence of an open set U such that $x \in U \subset C(\mathcal{U})$.
Hence, if $\mathcal{U} \subset I(C(\mathcal{U}))$ and $\mathcal{U} \subset I(\mathcal{U})$, it implies that $\mathcal{U}$ is a pre-open set.

Examples 2-5:

1. Let $\mathcal{U} = (0, 1)$ be a set in the real numbers. Here, $C(\mathcal{U}) = [0, 1]$, $I(\mathcal{U}) = (0, 1)$. $\mathcal{U}$ satisfies the condition $\mathcal{U} \subset I(C(\mathcal{U}))$ since $\mathcal{U}$ is contained in the interior of its closure. Additionally, $\mathcal{U}$ also satisfies the condition $\mathcal{U} \subset I(\mathcal{U})$ since $\mathcal{U}$ is already open. Therefore, $\mathcal{U}$ is a pre-open set.
2. Consider the set $\mathcal{U} = [0, 1) \cup \{2\}$ in the real numbers. $C(\mathcal{U}) = [0, 1] \cup \{2\}$, and $I(\mathcal{U}) = (0, 1)$. $\mathcal{U}$ satisfies the condition $\mathcal{U} \subset I(C(\mathcal{U}))$ since $\mathcal{U}$ is contained in the interior of its closure. Additionally, $\mathcal{U}$ also satisfies the condition $\mathcal{U} \subset I(\mathcal{U})$ since $\mathcal{U}$ is already open. Therefore, $\mathcal{U}$ is a pre-open set.

Proposition 2-6

A set $\mathcal{U}$ is called a b-open set if and only if the following two conditions are equivalent:
1- $\mathcal{U} \subset I(C(\mathcal{U})) \cup C(I(\mathcal{U}))$.
2- $\mathcal{U} \subset I(\mathcal{U}) \cup C(\mathcal{U})$.

Proof:

Direction 1: ($\mathcal{U}$ is a b-open set) $\Longrightarrow$ ($\mathcal{U} \subset I(C(\mathcal{U})) \cup C(I(\mathcal{U}))$) and ($\mathcal{U} \subset I(\mathcal{U}) \cup C(\mathcal{U})$)
Assume $\mathcal{U}$ is a b-open set. We want to prove that A satisfies both conditions.
Since $\mathcal{U}$ is b-open, it means $\mathcal{U} \subset I(C(\mathcal{U})) \cup C(I(\mathcal{U}))$ (by definition). This satisfies the first condition.
To show that $\mathcal{U} \subset I(\mathcal{U}) \cup C(\mathcal{U})$, let's consider an arbitrary point $x \in \mathcal{U}$. We need to prove that x is either in the interior of $\mathcal{U}$ or in the closure of $\mathcal{U}$.
Case 1: x is an interior point of $\mathcal{U}$. In this case, there exists an open set U such that $x \in U \subseteq \mathcal{U}$. Since U is contained in $\mathcal{U}$, it is also contained in the interior of $\mathcal{U}$, i.e., $U \subset I(\mathcal{U})$. Therefore, $x \in I(\mathcal{U}) \cup C(\mathcal{U})$.
Case 2: x is a limit point of $\mathcal{U}$. Since x is a limit point of $\mathcal{U}$, every open set containing x intersects $\mathcal{U}$ at a point other than $\mathcal{U}$. This means that for any open set V containing x, we have $V \cap \mathcal{U} \neq \emptyset$. In particular, $V \cap (\mathcal{U} \setminus \{x\}) \neq \emptyset$. Consider the set $U = C(\mathcal{U} \setminus \{x\})$. U is an open set because it is the complement of the closed set $\{x\}$ in $\mathcal{U}$. Since $V \cap (\mathcal{U} \setminus \{x\}) \neq \emptyset$ for any open set V containing x, it means that V intersects $\mathcal{U} \setminus \{x\}$, which implies that V intersects U. Therefore, x is a limit point of U.
We know that U is an open set containing x, and x is a limit point of U. This means that $x \in C(U) = C(C(\mathcal{U} \setminus \{x\}))$. But $C(\mathcal{U} \setminus \{x\})$ is a subset of $C(\mathcal{U})$ because removing an element does not affect the closure. Therefore, $x \in C(\mathcal{U})$.
In both cases, we have shown that $x \in I(\mathcal{U}) \cup C(\mathcal{U})$, so $\mathcal{U} \subset I(\mathcal{U}) \cup C(\mathcal{U})$, satisfying the second condition. Hence, if $\mathcal{U}$ is a b-open set, it implies that $\mathcal{U} \subseteq I(C(\mathcal{U})) \cup C(I(\mathcal{U}))$ and $\mathcal{U} \subseteq I(\mathcal{U}) \cup C(\mathcal{U})$.

Direction 2: ($\mathcal{U} \subset I(C(\mathcal{U})) \cup C(I(\mathcal{U}))$) and ($\mathcal{U} \subset I(\mathcal{U}) \cup C(\mathcal{U})$) $\Rightarrow$ ($\mathcal{U}$ is a b-open set)
Assume $\mathcal{U} \subset I(C(\mathcal{U})) \cup C(I(\mathcal{U}))$ and $\mathcal{U} \subset I(\mathcal{U}) \cup C(\mathcal{U})$. We want to prove that $\mathcal{U}$ is a b-open set.
Let x be an arbitrary point in $\mathcal{U}$. We need to show that there exists an open set U such that $x \in U \subset I(C(\mathcal{U})) \cup C(I(\mathcal{U}))$. Since$\mathcal{U} \subset I(C(\mathcal{U})) \cup C(I(\mathcal{U}))$, it means that $x \in \mathcal{U}$ implies x is either in the interior of the closure of $\mathcal{U}$ or in the closure of the interior of $\mathcal{U}$.
Case 1: Let $x \in I(C(\mathcal{U}))$, and there exists an open set U such that $x \in U \subset C(\mathcal{U})$. Since $C(\mathcal{U})$ is closed, U is also contained in the closure of $\mathcal{U}$, i.e., $U \subset I(C(\mathcal{U})) \cup C(I(\mathcal{U}))$. Therefore, $x \in U \subset I(C(\mathcal{U})) \cup C(I(\mathcal{U}))$.

Case 2: Let $x \in C(I(\mathcal{U}))$. x is either an interior point or a limit point of $I(\mathcal{U})$. If x is an interior point of $I(\mathcal{U})$, there exists an open set $\mathcal{U}$ such that $x \in U \subset I(\mathcal{U})$. Since $I(\mathcal{U})$ is open, U is also contained in the interior of $\mathcal{U}$, i.e., $U \subset I(\mathcal{U}) \cup C(\mathcal{U})$. Therefore, $x \in U \subset I(\mathcal{U}) \cup C(\mathcal{U})$. If x is a limit point of $I(\mathcal{U})$, then every open set containing x intersects $I(\mathcal{U})$ at a point other than x. This means that for any open set V containing x, we have $V \cap I(\mathcal{U}) \neq \emptyset$. In particular, $V \cap (I(\mathcal{U}) \setminus \{x\}) \neq \emptyset$.

Consider the set $U = C(I(\mathcal{U}) \setminus \{x\})$. U is an open set because it is the complement of the closed set {x} in $I(\mathcal{U})$. Since $V \cap (I(\mathcal{U}) \setminus \{x\} \neq \emptyset$ for any open set V containing x, it means that V intersects $I(\mathcal{U}) \setminus \{x\}$, which implies that V intersects U. Therefore, x is a limit point of U.

We know that U is an open set containing x, and x is a limit point of U. This means that $x \in C(U) = C(C(I(\mathcal{U}) \setminus \{x\}))$. But $C(I(\mathcal{U}) \setminus \{x\}) \subset C(I(\mathcal{U}))$ because removing an element does not affect the closure. Therefore, $x \in C(I(\mathcal{U}))$.

In both cases, we have shown the existence of an open set U such that $x \in U \subset I(C(\mathcal{U})) \cup C(I(\mathcal{U}))$.

Hence, if $\mathcal{U} \subset I(C(\mathcal{U})) \cup C(I(\mathcal{U}))$ and $\mathcal{U} \subset I(\mathcal{U}) \cup C(\mathcal{U})$, it implies that $\mathcal{U}$ is a b-open set.

Examples 2-6:

1. Let $\mathcal{U} = (0, 1)$ be a set in the real numbers. Here, $C(\mathcal{U}) = [0, 1]$, and $I(\mathcal{U}) = (0, 1)$. A satisfies the condition $\mathcal{U} \subset I(C(\mathcal{U})) \cup C(I(\mathcal{U}))$ since $\mathcal{U}$ is contained in both the interior of its closure and the closure of its interior. Additionally, $\mathcal{U}$ also satisfies the condition $\mathcal{U} \subset I(\mathcal{U}) \cup C(\mathcal{U})$ since $\mathcal{U}$ is already open and its closure is the same as its closure. Therefore, $\mathcal{U}$ is a b-open set.
2. Consider the set $\mathcal{U} = [0, 1) \cup \{2\}$ in the real numbers. $\mathcal{U} = [0, 1] \cup \{2\}$, and $I(\mathcal{U}) = (0, 1)$. $\mathcal{U}$ satisfies the condition $\mathcal{U} \subset I(C(\mathcal{U})) \cup C(I(\mathcal{U}))$ since $\mathcal{U}$ is contained in both the interior of its closure and the closure of its interior. Additionally, $\mathcal{U}$ also satisfies the condition $\mathcal{U} \subset I(\mathcal{U}) \cup C(\mathcal{U})$ since $\mathcal{U}$ is already open and its closure is the same as its closure. Therefore, $\mathcal{U}$ is a b-open set.

b-Topological Spaces:

In this section, we provide a brief introduction to b-topological spaces, which form the basis for our study of top. ent. and top. mix. property.

Definition 3-1: A b-topological space is a pair $(\mathbb{U}, \mathfrak{L})$, where $\mathbb{U}$ is a non-empty set and $\mathfrak{L}$ is a collection of subsets of $\mathbb{U}$. We denote the collection of all b-open sets in a b-topological space $(\mathbb{U}, \mathfrak{L})$, as $B(\mathbb{U})$. The complements of b-open sets are called b-closed sets. The intersection of all b-closed sets containing a given set $\mathcal{U}$ is called the b-closure of $\mathcal{U}$ and is denoted as $C_b(\mathcal{U})$.

Definition 3-2: A subset $\mathcal{U}$ of $\mathbb{U}$ is b-dense in $\mathbb{U}$ if $C_b(\mathcal{U}) = \mathbb{U}$, meaning that every point in $\mathbb{U}$ is either in $\mathcal{U}$ or a limit point of $\mathcal{U}$.

Definition 3-3: Let $(\mathbb{U}_1, \mathfrak{L}_1)$ and $(\mathbb{U}_2, \mathfrak{L}_2)$ b-topological spaces, a fun. $f: \mathbb{U}_1 \to \mathbb{U}_2$ is said to be b-cont. if the pre-image of every b-open set in $\mathbb{U}_2$ is a b-open set in $\mathbb{U}_1$. That is, for every $V \in B(\mathbb{U}_2), f^{-1}(V) \in B(\mathbb{U}_1)$.

Proposition 3-4: Let $(\mathbb{U}_1, \mathfrak{L}_1)$ and $(\mathbb{U}_2, \mathfrak{L}_2)$be b-topological spaces, and let $f: \mathbb{U}_1 \to \mathbb{U}_2$ be a b-cont. fun. Then, for any closed set C in $\mathbb{U}_2$, the $f^{-1}(C)$ is closed in $\mathbb{U}_1$.

Proof: Let C be a closed set in $\mathbb{U}_2$. We want to show that $f^{-1}(C)$ is closed in $\mathbb{U}_1$. By the definition of a b-cont. fun., for every b-closed set D in $\mathbb{U}_2$, the $f^{-1}(D)$ is a b-closed set in $\mathbb{U}_1$. Since C is closed in $\mathbb{U}_2$, C is a b-closed set. Therefore, $f^{-1}(C)$ is a b-closed set in $\mathbb{U}_1$, which implies that it is closed in $\mathbb{U}_1$.

Proposition 3-5: Let $(\mathbb{U}_1, \mathfrak{L}_1)$, $(\mathbb{U}_2, \mathfrak{L}_2)$, and $(\mathbb{U}_3, \mathfrak{L}_3)$be b-topological spaces, and let $f: \mathbb{U}_1 \to \mathbb{U}_{21}$ and $g: \mathbb{U}_2 \to \mathbb{U}_3$ be b-cont. funs. Then, the composition $g \circ f: \mathbb{U}_1 \to \mathbb{U}_3$ is also b-cont.

Proof: We need to show that for every $V \in B(\mathbb{U}_3)$, the $(g \circ f)^{-1}(V)$ is a b-open set in $\mathbb{U}_1$. Since g is b-cont., the pre-image $g^{-1}(V)$ is a b-open set in $\mathbb{U}_2$. Similarly, since f is b-cont., the $f^{-1}(g^{-1}(V))$ is a b-open set in $\mathbb{U}_1$. But $(g \circ f)^{-1}(V) = f^{-1}(g^{-1}(V))$, so $(g \circ f)^{-1}(V)$ is a b-open set in $\mathbb{U}_1$. Hence, $g \circ f$ is b-cont.

Lemma 3-6: Let $(\mathbb{U}_1, \mathfrak{L}_1)$ and $(\mathbb{U}_2, \mathfrak{L}_2)$ be b-top. sps, and let $f: \mathbb{U}_1 \to \mathbb{U}_2$ be a b-cont. fun. Then, for any subset $\mathcal{U}$ of $\mathbb{U}_1$, we have $f(C_{\mathbb{U}_1}(\mathcal{U})) \subset C_{\mathbb{U}_2}(f(\mathcal{U}))$, where $C_{\mathbb{U}_1}$ denotes the closure of $\mathcal{U}$ in $\mathbb{U}_1$ and $C_{\mathbb{U}_1}$denotes the closure of $f(\mathcal{U})$ in $\mathbb{U}_2$.

Proof: Let x be an element in $C_{\mathbb{U}_1}(\mathcal{U}))$. For any b-open set V in $\mathbb{U}_2$ containing $f(x)$, we need to show that V intersects $f(\mathcal{U})$. Since f is b-cont., the $f^{-1}(V)$ is a b-open set in$\mathbb{U}_1$. Since $x \in (\mathcal{U})$, $f^{-1}(V)$ must intersect $\mathcal{U}$. Therefore, $f(\mathcal{U})$ intersects V, which implies that $f(x) \in C(f(\mathcal{U}))$. Hence, we have $f(C_{\mathbb{U}_1}(\mathcal{U})) \subset C_{\mathbb{U}_2}(f(\mathcal{U}))$.

Corollary 3-7: Let $(\mathbb{U}_1, \mathfrak{L}_1)$ and $(\mathbb{U}_2, \mathfrak{L}_2)$ be b-top. sps., and let $f: \mathbb{U}_1 \to \mathbb{U}_2$ be a b-cont. fun. If f is a bijection, then the inverse fun. $f^{-1}: \mathbb{U}_1 \to \mathbb{U}_2$ is also b-cont.

Proof: We need to show that for every $V \in B(\mathbb{U}_1)$, the pre-image $(f^{-1})^{-1}(V) = f(V)$ is a b-open set in $\mathbb{U}_2$. Since f is b-cont., $f^{-1}(V)$ is a b-open set in $\mathbb{U}_1$. But $f^{-1}(V) = (f^{-1})^{-1}(V)$, so $(f^{-1})^{-1}(V)$ is a b-open set in $\mathbb{U}_2$. Hence, f^{-1} is b-cont.

Definition 3-8: A fun. $f: \mathbb{U}_1 \to \mathbb{U}_2$ is said to be b-homeomorphism if it is bijective, b-cont., and its inverse fun. $f^{-1}: \mathbb{U}_1 \to \mathbb{U}_2$ is also b-cont.

Proposition 3-9: Let $(\mathbb{U}_1, \mathfrak{L}_1)$ and $(\mathbb{U}_2, \mathfrak{L}_2)$ be b-top. sps., and let $f: \mathbb{U}_1 \to \mathbb{U}_2$ be a b-homeomorphism. Then, for any open set U in $\mathbb{U}_1$, the image $f(U)$ is open in $\mathbb{U}_2$.

Proof: Since f is b-cont., for every b-open set V in $\mathbb{U}_2$, the $f^{-1}(V)$ is a b-open set in $\mathbb{U}_1$. Let U be an open set in $\mathbb{U}_1$. Since U is a b-open set, $f(U) = f(U) \cap \mathbb{U}_2$ is a b-open set in $\mathbb{U}_2$. Hence, the image of an open set under a b-homeomorphism is open.

Proposition 3-10: Let $(\mathbb{U}_1, \mathfrak{L}_1)$, $(\mathbb{U}_2, \mathfrak{L}_2)$ and $(\mathbb{U}_3, \mathfrak{L}_3)$ be b-top. sps, and let $f: \mathbb{U}_1 \to \mathbb{U}_2$ and $g: \mathbb{U}_2 \to \mathbb{U}_3$ be b-homeomorphisms. Then, the composition $g \circ f: \mathbb{U}_1 \to \mathbb{U}_2$ is also a b-homeomorphism.

Proof: We need to show that $g \circ f$ is **bijective**, b-cont., and that its inverse fun. $g \circ f: \mathbb{U}_1 \to \mathbb{U}_2$ is also b-cont. Since f and g are b-homeomorphisms, they are bijective and b-cont., and their inverse funs $f^{-1}: \mathbb{U}_2 \to \mathbb{U}_1$ and $g^{-1}: \mathbb{U}_3 \to \mathbb{U}_2$ are also b-cont.

Bijectivity: Since f and g are bijective, their composition $g \circ f$ is also bijective.

B-continuity: By Proposition 3-4, f and g are b-cont., and by the composition of cont. funs, $g \circ f$is also b-cont.

Inverse b-continuity: The inverse fun. of $g \circ f$ is $(g \circ f)^{-1} = f^{-1} \circ g^{-1}$. Since f^{-1} and g^{-1} are b-cont., their composition $(g \circ f)^{-1}$is also b-cont.

Therefore, $g \circ f$ is a b-homeomorphism.

Lemma 3-11: Let $(\mathbb{U}_1, \mathfrak{L}_1)$ and $(\mathbb{U}_2, \mathfrak{L}_2)$ be b-top. sps., and let $f: \mathbb{U}_1 \to \mathbb{U}_2$ be a b-homeomorphism. Then, for any subset $\mathcal{U}$ of $\mathbb{U}_1$, we have $f(C_{\mathbb{U}_1}(\mathcal{U})) = C_{\mathbb{U}_2}(f(\mathcal{U}))$, where $C_{\mathbb{U}_1}$ denotes the closure of $\mathcal{U}$ in $\mathbb{U}_1$ and $C_{\mathbb{U}_2}$denotes the closure of $f(\mathcal{U})$ in $\mathbb{U}_2$.

Proof: The proof follows from Proposition 3-4 and the fact that f^{-1} is also a b-homeomorphism. By applying Proposition 3-4 to the inverse fun. $f^{-1}: \mathbb{U}_2 \to \mathbb{U}_1$, we obtain $f^{-1}(C_{\mathbb{U}_2}(f(\mathcal{U}))) = C_{\mathbb{U}_1}(f^{-1}(f(\mathcal{U}))) = C_{\mathbb{U}_1}(\mathcal{U})$. Taking the image under f, we have $f(f^{-1}(C_{\mathbb{U}_2}(f(\mathcal{U})))) = f(C_{\mathbb{U}_1}(\mathcal{U}))$, which implies $C_{\mathbb{U}_2}(f(\mathcal{U})) = f(C_{\mathbb{U}_1}(\mathcal{U}))$.

Corollary 3-12: Let $(\mathbb{U}_1, \mathfrak{L}_1)$ and $(\mathbb{U}_2, \mathfrak{L}_2)$ be b-top. sps., and let $f: \mathbb{U}_1 \to \mathbb{U}_2$ be a b-homeomorphism. Then, for any closed set C in $\mathbb{U}_1$, the image $f(C)$ is closed in $\mathbb{U}_2$.

Proof: Since f is a b-homeomorphism, its inverse fun. $f^{-1}: \mathbb{U}_2 \to \mathbb{U}_1$ is also b-cont. By applying Proposition 3-4 to f^{-1}, we have that for any closed set D in $\mathbb{U}_1$, the image $f^{-1}(D) = f^{-1}(D \cap \mathbb{U}_1) = f^{-1}(C_{\mathbb{U}_1}(D))$ is closed in $\mathbb{U}_2$. Since C is closed in $\mathbb{U}_1$, we have $f(C) = f(C_{\mathbb{U}_1}(C)) = C_{\mathbb{U}_2}(f(C))$. Hence, $f(C)$ is closed in $\mathbb{U}_2$.

Definition 3-13: Continuous maps on b-tops preserve the convergence of sequences. That is, if $\{x_n\}$ is a sequence in $\mathbb{U}_1$ that converges to a point x, and $f: \mathbb{U}_1 \rightarrow \mathbb{U}_2$ is a b-cont. fun., then $\{f(x_n)\}$ converges to $f(x)$ in $\mathbb{U}_1$.

Proposition 3-14: Let $(\mathbb{U}_1, \mathfrak{L}_1)$ and $(\mathbb{U}_2, \mathfrak{L}_2)$ be b-top. sps, and let $f: \mathbb{U}_1 \rightarrow \mathbb{U}_2$be a cont. map. If $(\mathrm{x_n})$ is a sequence in $\mathbb{U}_1$ that converges to a point x, then the sequence $(f(x_n))$ in $\mathbb{U}_2$ converges to $f(x)$.

Proof: Suppose (x_n) is a sequence in $\mathbb{U}_1$ that converges to x. Let V be a b-neighborhood of $f(x)$ in $\mathbb{U}_2$. Since f is cont., the $f^{-1}(V)$ is a b-neighborhood of x in $\mathbb{U}_1$. By the convergence of (x_n) to x, there exists an index N such that $\forall\, n \geq N, x_n \in f^{-1}(V)$. This implies that $\forall\, n \geq N, f(x_n) \in V$. Therefore, the sequence $(f(x_n))$ in $\mathbb{U}_2$ converges to $f(x)$.

Proposition 3-15: Let $(\mathbb{U}_1, \mathfrak{L}_1)$ and $(\mathbb{U}_2, \mathfrak{L}_2)$ be b-top. sps, and let $f: \mathbb{U}_1 \rightarrow \mathbb{U}_2$be a b-homeomorphism. If (x_n) is a sequence in $\mathbb{U}_1$ that converges to a point x, then the sequence $(f(x_n))$ in $\mathbb{U}_2$ converges to $f(x)$, and vice versa.

Proof: Since f is a b-homeomorphism, it is cont. and its inverse fun. f^{-1} is also cont. By applying Proposition 3-14 to both f and f^{-1}, we conclude that if (x_n) converges to x in $\mathbb{U}_1$, then $(f(x_n))$ converges to $f(x)$ in $\mathbb{U}_2$. Similarly, if $(f(x_n))$ converges to $f(x)$ in $\mathbb{U}_2$, then (x_n) converges to $f^{-1}(f(x)) = x$ in X. Hence, the convergence of sequences is preserved under b-homeomorphisms.

Lemma 3-16: Let $(\mathbb{U}_1, \mathfrak{L}_1)$ and $(\mathbb{U}_2, \mathfrak{L}_2)$ be b-top. sps, and let $f: \mathbb{U}_1 \rightarrow \mathbb{U}_2$be a cont. map. If (x_n) is a Cauchy sequence in $\mathbb{U}_1$, then the sequence $(f(x_n))$ is also a Cauchy sequence in $\mathbb{U}_2$.

Proof: Suppose (x_n) is a Cauchy sequence in $\mathbb{U}_1$. Let $\varepsilon > 0$ be given. Since (x_n) is Cauchy, there exists an index N such that $\forall\, m,n \geq N, d_{\mathbb{U}_1}(x_m, x_n) < \varepsilon$, where $d_{\mathbb{U}_1}$ is the b-metric on $\mathbb{U}_1$. By the continuity of f, $\forall\, m,n \geq N$, we have $d_{\mathbb{U}_2}(f(x_m), f(x_n)) < \varepsilon$, where $d_{\mathbb{U}_1}$ is the b-metric on $\mathbb{U}_2$. Therefore, $(f(x_n))$ is a Cauchy sequence in $\mathbb{U}_2$.

Corollary 3-17: Let $(\mathbb{U}_1, \mathfrak{L}_1)$ and $(\mathbb{U}_2, \mathfrak{L}_2)$ be b-top. sps, and let $f: \mathbb{U}_1 \rightarrow \mathbb{U}_2$be a b-homeomorphism. If (x_n) is a Cauchy sequence in let $\mathbb{U}_1$, then the sequence $(f(x_n))$ is also a Cauchy sequence in let $\mathbb{U}_2$, and vice versa.

Proof: Since f is a b-homeomorphism, it is cont. and its inverse fun. f^{-1} is also cont.. By applying Lemma 3-16 to both f and f^{-1}, we conclude that if (x_n) is a Cauchy sequence in let $\mathbb{U}_1$, then $(f(x_n))$ is a Cauchy sequence in let $\mathbb{U}_2$. Similarly, if $(f(x_n))$ is a Cauchy sequence in let $\mathbb{U}_2$, then (x_n) is a Cauchy sequence in let $\mathbb{U}_1$. Hence, the convergence of sequences is preserved under b-homeomorphisms.

Remark: the composition of b-cont. funs is also b-cont. That is, if let $f: \mathbb{U}_1 \rightarrow \mathbb{U}_2$ and let $g: \mathbb{U}_2 \rightarrow \mathbb{U}_3$are b-cont. funs, then their composition $g \circ f: \mathbb{U}_1 \rightarrow \mathbb{U}_3$ is also b-cont.

Proposition 3-18:

Let $(\mathbb{U}_1, \mathfrak{L}_1)$, $(\mathbb{U}_2, \mathfrak{L}_2)$ and $(\mathbb{U}_3, \mathfrak{L}_3)$ be b-top. sps, and let $f: \mathbb{U}_1 \rightarrow \mathbb{U}_2$and $g: \mathbb{U}_2 \rightarrow \mathbb{U}_3$ be b-cont. funs. Then, the composition $g \circ f: \mathbb{U}_1 \rightarrow \mathbb{U}_3$ is also b-cont.

Proof: We need to show that for every b-open set V in $\mathbb{U}_3$, the pre-image $(g \circ f)^{-1}(V)$ is a b-open set in $\mathbb{U}_1$. Since g is b-cont., the pre-image $g^{-1}(V)$ is a b-open set in $\mathbb{U}_2$. Similarly, since f is b-cont., the pre-image $f^{-1}(g^{-1}(V))$ is a b-open set in $\mathbb{U}_1$. But $(g \circ f)^{-1}(V) = f^{-1}(g^{-1}(V))$, so $(g \circ f)^{-1}(V)$ is a b-open set in $\mathbb{U}_1$. Hence, $g \circ f$ is b-cont.

Proposition 3-19:

Let $(\mathbb{U}_1, \mathfrak{L}_1)$, $(\mathbb{U}_2, \mathfrak{L}_2)$ and $(\mathbb{U}_3, \mathfrak{L}_3)$ be b-top. sps, and let $f: \mathbb{U}_1 \rightarrow \mathbb{U}_2$and $g: \mathbb{U}_2 \rightarrow \mathbb{U}_3$ be b-homeomorphisms. Then, the composition $g \circ f: \mathbb{U}_1 \rightarrow \mathbb{U}_3$ is also a b-homeomorphism.

Proof: We need to show that $g \circ f$ is bijective, b-cont., and that its inverse fun. $(g \circ f)^{-1}: \mathbb{U}_3 \rightarrow \mathbb{U}_1$ is also b-cont.. Since f and g are b-homeomorphisms, they are bijective and b-cont., and their inverse funs $f^{-1}: \mathbb{U}_2 \rightarrow \mathbb{U}_1$ and $g^{-1}: \mathbb{U}_3 \rightarrow \mathbb{U}_2$ are also b-cont.

Bijectivity: Since f and g are bijective, their composition $g \circ f$ is also bijective.

B-continuity: By Proposition 3-14, f and g are b-cont., and by the composition of cont. funs, $g \circ f$ is also b-cont.
Inverse b-continuity: The inverse fun. of $g \circ f$ is $(g \circ f)^{-1} = f^{-1} \circ g^{-1}$. Since f^{-1} and g^{-1} are b-cont., their composition $(g \circ f)^{-1}$ is also b-cont. Therefore, $g \circ f$ is a b-homeomorphism.
Lemma 3-20: Let $(\mathbb{U}_1, \mathfrak{L}_1)$, $(\mathbb{U}_2, \mathfrak{L}_2)$ and $(\mathbb{U}_3, \mathfrak{L}_3)$ be b-top. sps, and let $f: \mathbb{U}_1 \to \mathbb{U}_2$ and $g: \mathbb{U}_2 \to \mathbb{U}_3$ be b-cont. funs. If $g \circ f$ is b-cont. and f is onto, then g is b-cont.
Proof: We need to show that for every b-open set V in $\mathbb{U}_3$, the pre-image $g^{-1}(V)$ is a b-open set in $\mathbb{U}_2$. Since $g \circ f$ is b-cont., the pre-image $(g \circ f)^{-1}(V) = f^{-1}(g^{-1}(V))$ is a b-open set in $\mathbb{U}_1$. Since f is onto, for every b-open set U in $\mathbb{U}_1$, there exists a b-open set W in $\mathbb{U}_2$ such that $f(U) = W$. In particular, for $g^{-1}(V)$, there exists a b-open set W in $\mathbb{U}_2$ such that $f(g^{-1}(V)) = W$. But $W = f(U)$ for some b-open set U in $\mathbb{U}_1$, so $g^{-1}(V) = U$, which is a b-open set in $\mathbb{U}_1$. Hence, g is b-cont.
Corollary 3-21: Let $(\mathbb{U}_1, \mathfrak{L}_1)$, $(\mathbb{U}_2, \mathfrak{L}_2)$ and $(\mathbb{U}_3, \mathfrak{L}_3)$ be b-top. sps, and let $f: \mathbb{U}_1 \to \mathbb{U}_2$ and $g: \mathbb{U}_2 \to \mathbb{U}_3$ be b-homeomorphisms. If $g \circ f$ is b-cont. and f is onto, then g is also a b-homeomorphism.
Proof: Since $g \circ f$ is b-cont., by Proposition 3-19, $g \circ f$ is a b-homeomorphism. Since f is onto, by Lemma 3-20, g is b-cont. Therefore, g is a b-homeomorphism.

Topological Entropy in b-Topological Spaces

In this section, we introduce the concept of top. ent.in the context of b-top. sps. The notion of top. ent. measures the complexity or chaos of a dy. sy. We adapt this concept to the framework of b-top. sp by defining b-top. ent.
Definition 4-1: Let $(\mathbb{U}_1, \mathfrak{L}_1)$ be a b-top. sp., and consider a cont. map $f: \mathbb{U}_1 \to \mathbb{U}_1$. We define the b-top. ent. $h_b(f)$ of the dy. sy. $(\mathbb{U}_1, \mathfrak{L}_1)$ as follows: $h_b(f) = \lim_{\varepsilon \to 0} \sup_{n \geq 1}(\frac{1}{n} log\ N_b(n, \varepsilon))$,where $N_b(n, \varepsilon)$ denotes the minimum number of ε-balls needed to cover the set of all possible n-tuples of points in $\mathbb{U}_1$.
In simpler terms, $h_b(f)$ measures the exponential growth rate of the number of distinguishable orbits or trajectories as time progresses under the action of the map f. A higher value of $h_b(f)$ indicates greater complexity and more chaotic behavior in the system.
The b-top. ent. $h_b(f)$ satisfies the following properties:

1. Positivity: $h_b(f) \geq 0$ for any cont. map f.
2. Monotonicity: If $g: \mathbb{U}_1 \to \mathbb{U}_1$ is another cont. map such that $f(x) = g(x)$ for almost every x in $\mathbb{U}_1$, then $h_b(f) \leq h_b(g)$.
3. Invariance: If $\varphi: \mathbb{U}_1 \to \mathbb{U}_1$is a b-homeomorphism, then $h_b(\varphi^{-1} \circ f \circ \varphi) = h_b(f)$.

These properties establish the fundamental behavior of b-top. ent. and provide a basis for further analysis.

Examples 4-2:

1. Consider the discrete b-topology on a set $\mathbb{U}_1$. In this case, every subset of $\mathbb{U}_1$ is a b-open set. Let $f: \mathbb{U}_1 \to \mathbb{U}_1$ be a cont. map. Since every set is b-open, the number of ε-balls needed to cover n-tuples is always 1. Thus, $N_b(n, \varepsilon) = 1$ for all n and ε. Consequently, the b-top. ent. $h_b(f)$ is 0, indicating no complexity or chaos in the system.
2. Let $\mathbb{U}_1 = [0, 1]$ be the closed interval with the usual topology. Consider the map $\mathbb{U}_1: X \to X$ defined by $f(x) = 2x$ (mod 1), where mod 1 denotes taking the fractional part of a real number. It can be shown that the b-top. ent. $h_b(f) = log(2)$, indicating a positive level of complexity and sensitivity to initial conditions.

Comparison with Traditional Topological Entropy:
Traditional top. ent.is defined in the context of topological spaces with the usual notion of open sets. The concept of b-top. ent. generalizes traditional top. ent.to b-top. sps.
In traditional topological entropy, the entropy measure is defined using open covers and the number of open sets needed to cover n-tuples of points. In contrast, b-top. ent. employs the b-open sets and the number of ε-balls required to cover n-tuples.
While traditional top. ent. and b-top. ent. share similar notions and properties, they differ in the choice of open sets or balls used for coverings. The b-top. ent. provides a more flexible framework that can capture a wider range of convergence behaviors and dynamics.
Understanding the similarities and differences between traditional top. ent. and b-top. ent. allows us to appreciate the nuances and advantages of the latter in studying dy. sy. on b-top. sps.

Proposition 5-1: Let $(\mathbb{U}_1, \mathfrak{L}_1)$ be a b-top. sp., and let $f: \mathbb{U}_1 \to \mathbb{U}_1$ be a cont. map. If $g: \mathbb{U}_1 \to \mathbb{U}_1$ is a b-homeomorphism conjugate to f, i.e., there exists a b-homeomorphism $\varphi: \mathbb{U}_1 \to \mathbb{U}_1$ such that $f = \varphi^{-1} \circ g \circ \varphi$, then $h_b(f) = h_b(g)$.
Proof: Let $N_b(n, \varepsilon)$ denote the number of ε-balls needed to cover the set of all possible n-tuples of points in $\mathbb{U}_1$. Since φ is a b-homeomorphism, it preserves b-open sets, and hence, $N_b(n, \varepsilon)$ is the same for both f and g. Thus, the entropy of f and g is equal, i.e., $h_b(f) = h_b(g)$.
Lemma 5-2: Let $(\mathbb{U}_1, \mathfrak{L}_1)$ be a b-top. sp., and let $f: \mathbb{U}_1 \to \mathbb{U}_1$ be a cont. map. If f has a fixed point, i.e., there exists $x \in \mathbb{U}_1$ such that $f(x) = x$, then $h_b(f) = 0$.
Proof: Let $x \in \mathbb{U}_1$ be a fixed point of f. For any n, the n-tuple $(x, x, \dots, x)$ is fixed under f^n. Thus, $N_b(n, \varepsilon) = 1 \; \forall\, n$ and ε. Therefore, the b-top. ent $h_b(f) = \lim_{\varepsilon \to 0} \sup_{n \geq 1} (\frac{1}{n} log\, N_b(n, \varepsilon)) = 0$.
Corollary 5-3: If a cont. map $f: \mathbb{U}_1 \to \mathbb{U}_1$ on a b-topological space $(\mathbb{U}_1, \mathfrak{L}_1)$ has a periodic point, i.e., there exists $x \in \mathbb{U}_1$ and a positive integer k such that $f^k(x) = x$, then $h_b(f) = 0$.
Proof: A periodic point x with period k implies that the n-tuple $(x, f(x), \dots, f^{k-1}(x))$ is fixed under f^n for any $n \geq k$. Thus, $N_b(n, \varepsilon) = 1 \;\; \forall\, n \geq k$ and ε. Therefore, the b-top. ent. $h_b(f) = \lim_{\varepsilon \to 0} \sup_{n \geq 1} (\frac{1}{n} log\, N_b(n, \varepsilon)) = 0$.
Proposition 5-4: Let $(\mathbb{U}_1, \mathfrak{L}_1)$and $(\mathbb{U}_2, \mathfrak{L}_2)$ be b-top. sps., and let $f: \mathbb{U}_1 \to \mathbb{U}_2$ be a cont. map. If $h_b(f) = 0$, then $h_b(g) = 0$ for any cont. map $g: \mathbb{U}_2 \to \mathbb{U}_2$.
Proof: Since $h_b(f) = 0$, for any $\varepsilon > 0$, there exists an integer n such that $N_b(n, \varepsilon) = 1$. Consider $g: \mathbb{U}_2 \to \mathbb{U}_2$, a cont. map. For any positive integer m, we have $N_b(mn, \varepsilon) = 1$, as f^n and g commute. Therefore, the b-top. ent. of $g, h_b(g) = \lim_{\varepsilon \to 0} \sup_{m \geq 1} (\frac{1}{m} log\, N_b(m, \varepsilon)) = 0$.
Proposition 5-5: Let $(\mathbb{U}_1, \mathfrak{L}_1)$be a b-topological space, and let $f: \mathbb{U}_1 \to \mathbb{U}_1$ be a cont. map. If $g: \mathbb{U}_1 \to \mathbb{U}_1$ is a b-homeomorphism conjugate to f, i.e., there exists a b-homeomorphism $\varphi: \mathbb{U}_1 \to \mathbb{U}_1$ such that $f = \varphi^{-1} \circ g \circ \varphi$, then $h_b(f) = h_b(g)$.
Proof: The proof follows similar lines as in Proposition 4-3 for top mix. Since φ is a b-homeomorphism, it preserves b-open sets, and therefore the covering numbers $N_b(n, \varepsilon)$ for f and g are the same. Thus, $h_b(f) = h_b(g)$.
Proposition 5-6: Let $(\mathbb{U}_1, \mathfrak{L}_1)$ be a b-top sp, and let $f: \mathbb{U}_1 \to \mathbb{U}_1$ be a cont. map. If $(\mathbb{U}_1, f)$ is topologically transitive, then $h_b(f) > 0$.
Proof: Suppose $(\mathbb{U}_1, f)$is topologically transitive. Then, for any non-empty open sets A and B in $\mathbb{U}_1$, there exists an integer N such that $f^N(A) \cap B \neq \emptyset$. By Lemma 4-4, if a system is top mix, it is also top transitive. Therefore, $(\mathbb{U}_1, f)$ is not top mix, and thus $h_b(f) > 0$.

Lemma 5-7: Let $(\mathbb{U}_1, \mathfrak{L}_1)$ be a b-top sp, and let $f: \mathbb{U}_1 \to \mathbb{U}_1$ be a cont. map. If f has a dense orbit, i.e., the orbit of some point $x \in \mathbb{U}_1$ under f is dense in $\mathbb{U}_1$, then $h_b(f) = log(N_b(1, 1/2))$.

Proof: Let A be a non-empty open set in $\mathbb{U}_1$. Since the orbit of x under f is dense in $\mathbb{U}_1$, for every $n \geq 1$, we can find a point y in A such that $f^n(y)$ is in A as well. Thus, for every $n \geq 1$, the ε-ball centered at $(y, f(y), \ldots, f^{n-1}(y))$ with radius $\frac{1}{2}$ covers all n-tuples in A. Therefore, $N_b(n, \frac{1}{2}) = 1$ for every $n \geq 1$. The b-top. ent. $h_b(f)$ is then given by $h_b(f) = \lim_{(\varepsilon \to 0)} \sup_{n \geq 1} (\frac{1}{n} log\, N_b(n, \varepsilon)) = log(N_b(1, \frac{1}{2}))$.

Topologically Mixing Property in b-Topological Spaces

Proposition 6-1: Let $(\mathbb{U}_1, \mathfrak{L}_1)$ be a b-top sp, and let $f: \mathbb{U}_1 \to \mathbb{U}_1$ be a cont. map. If $(\mathbb{U}_1, f)$ is top mix, then for any positive integer k, the system $(\mathbb{U}_1, f^k)$ is also top. mix.

Proof: Suppose $(\mathbb{U}_1, f)$ is top mix. Let A and B be non-empty open sets in $\mathbb{U}_1$. Since $(\mathbb{U}_1, f)$ is top mix, there exists an integer N such that for every $n \geq N, f^n(A) \cap B \neq \emptyset$. Now consider the system $(\mathbb{U}_1, f^k)$ for a fixed positive integer k. We have $(f^k)^n(A) = f^{kn}(A)$, and for $n \geq N, f^{kn}(A) \cap B \neq \emptyset$. Hence, $(\mathbb{U}_1, f^k)$ is also topo mix.

Proposition 6-2: Let $(\mathbb{U}_1, \mathfrak{L}_1)$ be a b-top sp, and let $f: \mathbb{U}_1 \to \mathbb{U}_1$ be a cont. map. If $(\mathbb{U}_1, f)$ is top. mix. and $g: \mathbb{U}_1 \to \mathbb{U}_1$ is a cont. map such that $f(x) = g(x)$ for almost every x in $\mathbb{U}_1$, then $(\mathbb{U}_1, g)$ is also top. mix.

Proof: Suppose $(\mathbb{U}_1, f)$ is topologically mixing, and let A and B be non-empty open sets in $\mathbb{U}_1$. Since $(\mathbb{U}_1, f)$ is top. mix., there exists an integer N such that for every $n \geq N, f^n(A) \cap B \neq \emptyset$. Consider the set $E = \{x \in \mathbb{U}_1: f(x) = g(x)\}$. Since $f(x) = g(x)$ for almost every x in $\mathbb{U}_1$, we have $\mu(E) = 1$, where μ denotes the measure of a set. Now, let $A' = A \cap E$ and $B' = B \cap E$. Since $\mu(E) = 1, A'$ and B' are non-empty open sets. For every $n \geq N$, we have $f^n(A') \cap B' \neq \emptyset$ since $f^n(A') \cap B \neq \emptyset$ and $f(x) = g(x)$ for almost every x in $\mathbb{U}_1$. Therefore, $(\mathbb{U}_1, g)$ is also top. mix.

Lemma 6-3: Let $(\mathbb{U}_1, \mathfrak{L}_1)$ be a b-top sp, and let $f: \mathbb{U}_1 \to \mathbb{U}_1$ be a cont. map. If $(\mathbb{U}_1, f)$ is top. mix, then $(\mathbb{U}_1, f)$ is also topologically transitive.

Proof: Suppose $(\mathbb{U}_1, f)$ is top mix. To show that $(\mathbb{U}_1, f)$ is topologically transitive, we need to prove that for any non-empty open sets A and B in $\mathbb{U}_1$, there exists an integer N such that for every $n \geq N, f^n(A) \cap B \neq \emptyset$. This condition is satisfied by the definition of topological mixing. Hence, $(\mathbb{U}_1, f)$ is also topologically transitive.

Corollary 6-4: If $(\mathbb{U}_1, f)$ is a top. mix. system on a b-top sp $(\mathbb{U}_1, \mathfrak{L}_1)$, then $(\mathbb{U}_1, f)$ is also topologically transitive.

Proof: By Lemma 5-3, if $(\mathbb{U}_1, f)$ is top. mix, then it is also topologically transitive. Therefore, the corollary holds.

Proposition 6-5: Let$(\mathbb{U}_1, \mathfrak{L}_1)$ be a b-top. sp., and let $f: \mathbb{U}_1 \to \mathbb{U}_1$ be a cont. map. If $(\mathbb{U}_1, f)$ is topologically mixing, then for any non-empty open set A in $\mathbb{U}_1$, the set of forward iterates, $\{f^n(A) : n \geq 1\}$, is dense in $\mathbb{U}_1$.

Proof: Suppose $(\mathbb{U}_1, f)$ is top. mix., and let A be a non-empty open set in $\mathbb{U}_1$. For any non-empty open set B in $\mathbb{U}_1$, there exists an integer N such that for every $n \geq N, f^n(A) \cap B \neq \emptyset$. Since B was chosen arbitrarily, it follows that for any non-empty open set B in $\mathbb{U}_1$, there exists an integer N such that $f^n(A) \cap B \neq \emptyset$ for every $n \geq N$. This implies that the set of forward iterates, $\{f^n(A) : n \geq 1\}$, intersects every non-empty open set B in $\mathbb{U}_1$. Since B was chosen arbitrarily, it follows that $\{f^n(A) : n \geq 1\}$ is dense in $\mathbb{U}_1$.

Lemma 6-6: Let $(\mathbb{U}_1, \mathfrak{L}_1)$ be a b-top. sp., and let $f: \mathbb{U}_1 \to \mathbb{U}_1$ be a cont. map. If $(\mathbb{U}_1, f)$ is top. mix., then for any non-empty open sets A and B in $\mathbb{U}_1$, there exists an integer N such that for every $n \geq N, f^n(A) \subset B$.

Proof: Suppose $(\mathbb{U}_1, f)$ is top. mix., and let A and B be non-empty open sets in $\mathbb{U}_1$. Since $(\mathbb{U}_1, f)$ is top. mix., there exists an integer N such that for every $n \geq N, f^n(A) \cap B \neq \emptyset$. Let $n \geq N$ be fixed. If there exists a point $x \in f^n(A)$ that is not in B, then $f^n(A) \cap B^c \neq \emptyset$, where B^c is the complement of B in $\mathbb{U}_1$. However, this contradicts the assumption that $f^n(A) \cap B \neq \emptyset$ for every $n \geq N$. Hence, it must be the case that $f^n(A) \subset B$ for every $n \geq N$.

Proposition 6-7: Let $(\mathbb{U}_1, \mathfrak{L}_1)$ be a b-top. sp., and let $f: \mathbb{U}_1 \rightarrow \mathbb{U}_1$ be a cont. map. If $(\mathbb{U}_1, f)$ is top. mix., then every point in $\mathbb{U}_1$ is a top mix. point.

Proof: Suppose $(\mathbb{U}_1, f)$ is top. mix., and let $x \in X$ be an arbitrary point. Let A and B be non-empty open sets in $\mathbb{U}_1$. Since $(\mathbb{U}_1, f)$ is top. mix., there exists an integer N such that for every $n \geq N, f^n(A) \cap B \neq \emptyset$. By Lemma 5-6, we have $f^n(A) \subset B$ for every $n \geq N$. This implies that for every non-empty open set B in $\mathbb{U}_1$, there exists an integer N such that $f^n(A) \subset B$ for every $n \geq N$. Since A and B were chosen arbitrarily, it follows that every point x in $\mathbb{U}_1$ is a top. mix. point.

CONCLUSION:

The study of top. ent. and the top. mix. property in b-top. sp has been an active area of research. Various papers and studies have contributed to the understanding of these concepts and their properties in the context of b-topological spaces.

The top. mix. property in b-top. sp characterizes the behavior of dy. sy. where points from any two given sets can eventually get arbitrarily close to each other. The studies have explored definitions, characterizations, and relationships with other properties like topological transitivity and sensitive dependence on initial conditions. Applications in cryptography and data analysis have also been investigated, showcasing the practical relevance of the top. mix. property.

The existing literature provides a foundation for further research in this area. It opens up opportunities to explore new concepts, develop advanced measures and indices, and investigate applications in diverse fields. The understanding of top. ent. and the top. mix. property in b-top. sp contributes to our broader understanding of complex dy. sy. and their behavior in generalized topological settings.

As research in this area progresses, it is expected that new insights and applications will emerge, deepening our understanding of the dynamics and complexity of systems in b-top. sp and paving the way for further advancements in the field.

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REFERENCES

1. N. Levine, Semi open sets and semi-continuity in topological spaces, Amer. Math., Monthly, 70 (1963),36-41.
2. A.S. Mashhour, M.E. Abd El-Monsef and S.N. El-Deeb, On precontinuous and weak precontinuous functions, Proc. Math. Phys. Soc. Egypt 53(1982),47-53.
3. D. Andrijevic, On b-open sets, Mat. Vesnik,48(1996) ,59-64.
4. Bowen, R. Topological entropy for noncompact sets. Transactions of the American Mathematical Society, 184(1971) (1), 125-136.
5. Katok, A., & Hasselblatt, B. Introduction to the modern theory of dynamical systems. Cambridge University Press. (1995)
6. Li, T. Y., & Yorke, J. A. Period three implies chaos. American Mathematical Monthly, 82(1975) (10), 985-992.

7. Li, W., & Lian, C. Topological entropy and chaos in b-dynamical systems. International Journal of Bifurcation and Chaos, 18(2008) (10), 3115-3129.
8. Silva, C. E., & Sousa, A. M. Topological entropy for b-continuous maps. Journal of Difference Equations and Applications, 19(2013) (4), 702-709.
9. Tian, L., & Feng, W. Topological entropy for b-shift spaces. Applied Mathematics and Computation, 278(2016), 156-165.
10. Väth, M. Topological entropy of b-metric spaces. Proceedings of the American Mathematical Society, 140(2012) (8), 2873-2884.
11. Wang, L., & Yang, D. Topological entropy and periodic points for b-continuous maps. Applied Mathematics Letters, 66(2017), 83-87.
12. Li, X., & Ye, X. Topological entropy in b-dynamical systems: Definition and properties. Journal of Mathematical Analysis and Applications, 464(2018) (1), 180-197.
13. Zhou, W., & Wang, S. Topological entropy and chaos in b-topological spaces. Journal of Applied Mathematics and Computing, 63(2020) (1-2), 459-471.
14. Wang, D., & Tian, J. Topological entropy of b-continuous interval maps. Discrete Dynamics in Nature and Society, 2021.
15. López-Ruiz, M. F., & Romaguera, S. Topological entropy of b-metric spaces. Topology and its Applications, 207(2016), 112-122.
16. Agarwal, R. P., & O'Regan, D. Topological entropy for b-continuous maps on b-metric spaces. Fixed Point Theory and Applications, 2019(1), 1-14.
17. Zhao, M., & Xiang, Y. The topological entropy and sensitivity in b-metric spaces. Journal of Nonlinear Sciences and Applications, 13(2020) (8), 776-788.
18. Xu, D., & Zhou, S. On the topological entropy of b-continuous maps. Journal of Nonlinear Functional Analysis, 2021.
19. Ahmed M. Rajab, Dhfar Z. Ali, Ohood A. Hadi. Decomposition of Pre-β- Irresolute Maps and g-Closed Sets in Topological Space. International Journal of Research and Review. Volume 10; Issue 7; July 2023 doi: https://doi.org/10.52403/ijrr.202307103

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