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Patterns and Line-Adjacency Matrix Strategy on a Game of Induced Matching

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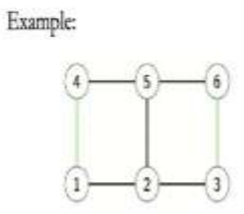
ABSTRACT

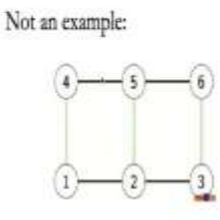
A matching is an independent set of edges in a graph G; an induced matching is a matching with an additional property that no two of its edges are joined by an edge in G. An induced matching M in a graph G is maximal if no other induced matching in G contains M. In the proposed game, players alternate choose edges on a graph while maintaining an induced matching. They continue until the matching is maximal (i.e. no player can choose another edge), and the last player to choose an edge wins. This paper will discuss patterns for this game and prove results on several categories of graphs including Complete Graphs, Path Graphs, Cycle Graphs, and Ladder Graphs. In addition, a novel Line-Adjacency Matrix method has been defined and proved to calculate all possible edges for the next move in a connected graph.

Keywords: Induced Matching, Adjacency Matrix, Line-Adjacency Graph

INTRODUCTION

An induced matching in a graph can be defined as a matching that arises as a result of an induced subgraph within the graph. It provides a framework for finding optimal pairings or matchings within a graph, where the pairs are determined based on certain predefined criteria or constraints [2].





The concept of induced matching finds applications in various domains, including computer science, operations research, and social network analysis. It is often utilized in solving optimization problems, designing efficient algorithms, and analyzing network structures.

I investigated the maximum and minimum moves in the Game of Induced Matching originally introduced by Francis Su [5]. The game contains the follows rules:

1. Two players take turns selecting an edge while maintaining an induced matching.
2. Once an edge is selected, all edges that are at most distance-1 adjacent are covered or "eliminated".
3. If there are no more available edges remaining (the matching is maximal), then the game ends.
4. The player to select the last edge wins. For this problem you will explore the "matching game" for various graphs and graph families. One potential question to explore would be whether there are graphs without a pre-determined outcome and whether a certain player has a winning strategy

1.An Algorithm for Available Moves with Adjacency Matrix

1.1 Line-Adjacency Matrix

Let G be the starting graph of the game. Construct L(G), the line graph of G. Now, draw an adjacency matrix of L(G); we will define this as the Line-Adjacency matrix of graph G and denote this as LA(G).

1.2 Theorem 1: Line-Adjacency Matrix finds the possible edges for the next move in a connected graph

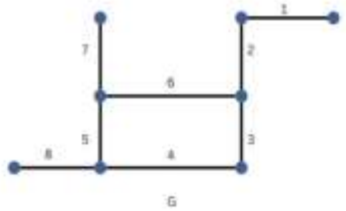
Proof: Let G be the connected starting graph of the game. Let V, E be the number of vertices and edges in G. Let L(G) be the line graph of G and LA(G) be the Line-Adjacency matrix of graph G as defined previously. Let i, j be some arbitrary vertices in L(G) with corresponding edges in G. Since L(G) is a non-directed graph, the values of the entry $LA(G)_{i,j}$ shows the number of edges between vertices i and j in L(G). The values of the entry $LA^2(G)_{i,j}$ (G) shows the number of different walks of length 2 between vertices i and j in L(G)[4]. Thus, the entries in $LA(G) + LA^2(G)$ represents the number of walks length 1 and 2 connecting two vertices in L(G).

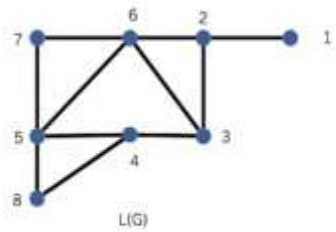
In the line graph L(G), for each edge in G there is a corresponding vertex in L(G). Thus, an edge between vertices i and j in L(G) corresponds to edges i and j in G being incident to a common vertex. Thus, the entries of $LA(G) + LA^2(G)$ represents the number of adjacent edges and edges joined by an edge in G [3]. Therefore, the 0 entries in $LA(G) + LA^2(G)$ shows that the corresponding edge i and j are neither adjacent to each other nor joined by an edge.

In an induced matching, the independent set of edges in graph G must have no two edges adjacent and no two edges joined by an edge [1]. Since G is connected, there must be at least one path connecting edges i and j. Therefore, the 0 of $LA(G) + LA^2(G)$ represents that there must be a walk greater than length 2 connecting edges i and j. This means that edge j is a possible next move if edge i is taken. If $LA(G)_{i,j} + LA^2(G)_{i,j} = 0$, then adding rows i and j gives directions for the possible edges for the third move. The common zeros in rows i and j provide us with the edges that are neither adjacent to i or j nor joined by an edge to i or j. Thus, if $LA(G)_{i,j} + LA^2(G)_{i,j} = 0$ and $LA(G)_{i,k} + LA^2(G)_{i,j} + LA(G)_{j,k} + LA^2(G)_{j,k} = 0$, then k is an available edge for the next move if i and j are taken. Without loss of generality, the corresponding column of 0s in the sum of the previous rows of $LA(G) + LA^2(G)$ always shows the next possible moves.

1.3 An Application Example

Graph G and construct the line graph L(G) are shown below:





Compute LA(G), the adjacency matrix of L(G):

	1	2	3	4	5	6	7	8
1	0	1	0	0	0	0	0	0
2	1	0	1	0	0	1	0	0
3	0	1	0	1	0	1	0	0
4	0	0	1	0	1	0	0	1
5	0	0	0	1	0	1	1	1
6	0	1	1	0	1	0	1	0
7	0	0	0	0	1	1	0	0
8	0	0	0	1	1	0	0	0

Compute $LA^2(G)$:

	1	2	3	4	5	6	7	8
1	1	0	1	0	0	1	0	0
2	0	3	1	1	1	1	1	0
3	1	1	3	0	2	1	1	1
4	0	1	0	3	1	2	1	1
5	0	1	2	1	4	1	1	1
6	1	1	1	2	1	4	1	1
7	0	1	1	1	1	1	2	1
8	0	0	1	1	1	1		2

$LA(G) + LA^2(G)$=

	1	2	3	4	5	6	7	8
1	1	0	1	0	0	1	0	0
2	0	3	1	1	1	1	1	0
3	1	1	3	0	2	1	1	1
4	0	1	0	3	1	2	1	1
5	0	1	2	1	4	1	1	1
6	1	1	1	2	1	4	1	1
7	0	1	1	1	1	1	2	1
8	0	0	1	1	1	1		2

Thus, we know for example that if the first move is on edge 1, then edges 4, 5, 7, 8 are available for the next move. Add row 1 and row 4: (1 2 2 3 2 3 1 2). Note there is no 0 entry, thus there is no additional move available, the game is complete. Similarity, the game ends if the 2nd move is 5 or 7 or 8. Thus a winning strategy for player 1 is to move to edge 3 and 6, immediately ending the game. If player 1 moves to edge 1, 2, 4, 5, 7, 8, then player 2 wins.

1.4 Theorem 2: If G contains multiple connected components, then the maximum number of moves is equal to the sum of the maximum number of moves in each connected component, and the minimum number of moves is equal to the sum of the minimum number of moves in each connected component.

Proof: If G contains multiple connected components, each component can be considered its own game of matching. The maximum number of moves and the minimum number of moves of each connected component can be found independently. The minimum number of moves for the game to complete is equal to the sum of the minimum number of moves of all connected components. The maximum number of moves for the game to complete is equal to the sum of the maximum number of moves of all connected components.

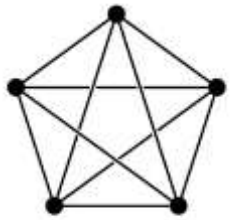
2. Investigating Maximum and Minimum Number of Moves in Simple Graphs

2.1 Complete Graphs

In a complete graph, there is only 1 move available.

Proof: Let K_n be a complete graph on n vertices. Create a line-adjacency matrix $LA(K_n)$. By definition, each vertex is adjacent to all n-1 other vertices in K_n by exactly one edge. Thus, $LA(K_n)$ contains a main diagonal of 0, and 1 in all other entries. Through matrix multiplication, $LA^2(K_n)$ contains a main diagonal of n-1 and n-2 in all other entries. Thus, all entries in $LA(K_n) + LA^2(K_n)$ are non-zero. By Theorem 1, after the initial move, there will be no next move possible.

2.1.1 An Example on LA(K_5):



K_5

LA(K_5)=

0	1	1	1	1
1	0	1	1	1
1	1	0	1	1
1	1	1	0	1
1	1	1	1	0

LA^2(K_5)=

4	3	3	3	3
3	4	3	3	3
3	3	4	3	3
3	3	3	4	3
3	3	3	3	4

LA(K_5)+ LA^2(K_5)=

5	4	4	4	4
4	5	4	4	4
4	4	5	4	4
4	4	4	5	4
4	4	4	4	5

LA(K_5) + LA^2(K_5) contains no 0 entries, thus once the first move is made, there are no edges available for the next move.

2.2 Theorem 3: The minimum number of edges in the induced matching set of a path graph is ⌈E/5⌉ where E is the number of edges in the graph and ⌈x⌉ represents the smallest integer greater than or equal to x.

Proof: By definition, a path graph is a graph that can be drawn so that all its vertices and edges lie on a single straight line. Without loss of generality, assume the graph is drawn left to right in a straight line. Then, selecting any edge for the induced matching set will at most eliminate 5 edges: the edge itself, two edges to the left, and 2 edges to the right.

To achieve the minimum number of edges, each edge selected should eliminate the most number of edges possible. If E is divisible by 5, then the minimum number of edges is E/5. Note that this can also be expressed as ⌈E/5⌉. If E is not divisible by 5 then the following can be applied:

Let n be the largest multiple of 5 less than E. Then n = 5k where k is some positive integer. Then E = 5k + b where b ∈ 1, 2, 3, 4. The minimum number of edges in the induced matching set for the first n edges contains k edges as stated previously. Since b < 5, only one edge is needed to eliminate all remaining b edges. Thus, the minimum number of edges in the induced matching set for the whole graph is k + 1, which is ⌈E/5⌉.

Since E must either be a multiple of 5 or not, and both cases can be expressed as $\lceil E/5 \rceil$., the minimum number of edges in the induced matching set of a path graph is $\lceil E/5 \rceil$.

2.3 Path Graphs

2.3.1 The minimum number of vertices in the induced matching set of a path graph is $\lceil V - 15 \rceil$ where V is the number of vertices in the graph.

Proof: In a path graph, E = V-1. Thus, this result follows from Theorem 3.

2.3.2 The maximum number of edges in the induced matching set of a path graph is $\lceil E/3 \rceil$ where E is the number of edges in the graph.

Proof: Assume the path graph is drawn left to right in a straight line. By choosing the leftmost edge, only 3 edges are eliminated. Then, the leftmost edge that would still maintain an induced matching can be chosen every subsequent turn.

Thus, every chosen edge except the last one would eliminate 3 edges, and the last chosen edge would eliminate either 1, 2, or 3 edges. By using the same logic as Theorem 3, but with 3 instead of 5, the maximum number of edges in the induced matching set for the whole graph is $\lceil E/3 \rceil$.

The maximum number of vertices in the induced matching set of a path graph is $\lceil V - 13 \rceil$ where V is the number of vertices in the graph. Proof: Based on the definition of a path graph, E = V-1. Thus, this result follows from Theorem 4.

2.4 Cycle Graphs

2.4.1 The minimum number of edges in the induced matching set of a cycle graph is $\lceil E/5 \rceil$ where E is the number of edges in the graph; the minimum number of vertices in the induced matching set of a cycle graph is $\lceil V/5 \rceil$ where V is the number of vertices in the graph

Proof: Based on Theorem 3, each edge chosen except for the last one will eliminate 5 edges and the result is $\lceil E/5 \rceil$. In a cycle graph, V = E. The result then follows from the previous conjecture $\lceil V/5 \rceil$.

2.4.2 The maximum number of edges in the induced matching set of a cycle graph is $\lfloor E/5 \rfloor$ where E is the number of edges in the graph and $\lfloor x \rfloor$ represents the largest integer less than or equal to x.

Proof: When E < 5, the maximum number of edges in the induced match-ing set is 1. When E ≥ 5, the first edge selected will always remove 5 edges. Then, the remaining edges can be treated as a path graph since the two edge vertices are not adjacent. Thus, by Theorem 4, the maximum number of edges in the induced matching set for the cycle graph would be $\lceil \frac{E-5}{3} \rceil + 1$, which is equivalent to $\lfloor E/3 \rfloor$.

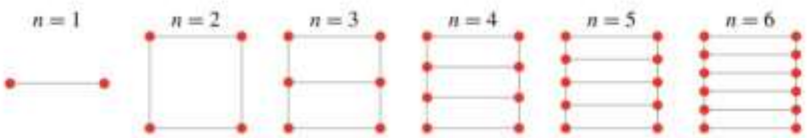
2.4.3 The maximum number of vertices in the induced matching set of a cycle graph is $\lfloor V/3 \rfloor$ where V is the number of vertices in the graph.

Proof: In a cycle graph, V = E. Thus, by the previous conjecture, the maximum number of vertices in the induced matching set of a cycle graph is $\lfloor V/3 \rfloor$.

2.5 Ladder Graph

2.5.1 In a ladder graph L_n, E=3n-2, where E is the number of edges.

Let Ln be a ladder graph, let E be the number of edges in Ln. Ln consists of two paths each with n vertices with an additional edge connecting each pair of corresponding vertices between the two paths. Examples from L1 to L6 are shown below:



As proven in the section of Path Graphs, a path graph on n vertices has n-1 edges. By the definition of ladder graph, Ln consists of two paths each with n vertices with an additional edge connecting each pair of corresponding vertices between the two paths. Thus, n additional edges are required to connect each pair of corresponding vertices on two paths with n vertices. Therefore, E=2(n-1)+n=3n-2.

2.5. Maximum Moves in a Ladder Graph L_n is ⌈n/2⌉

Proof: Case 1 (L1, L2): L1 and L2 have a maximum of 1 move, which is ⌈n/2⌉. The ladder L1 consists of a single rung. To eliminate all the edges, only one move is required. The ladder L2 consists of two rungs. By choosing the top rung, all the edges can be eliminated in one move. Therefore, the maximum number of moves for L1 and L2 is ⌈n/2 ⌉.

Case2 (Ln, where $n \geq 2$): Let's consider the ladder L3. Starting from the top rung, not including itself, choosing the top edge eliminates 5 edges. Choosing one of the two top side edges would eliminate 7 edges. Thus, to maximize the number of moves, we want to eliminate the least number of edges every turn. Therefore, it is most optimal to eliminate the top edge. Continuing this optimal pattern by choosing every subsequent available center rung would give us the maximum number of edges eliminated.

Eventually, the edges in the induced matching set are every other center rung. The number of center rungs is n. For odd n, choosing every other rung would yield (n+1)/2 selected edges, which is ⌈n/2⌉. For even n, choosing every other rung would yield exactly n/2 edges, which is ⌈n/2 ⌉. Therefore, the maximum number of moves for L3 and L4 is 2.

Hence, we have proved that for larger ladders with size n, the maximum number of moves is ⌈n/2⌉.

DISCUSSION

This paper provides an overview of patterns generalizations for the game of induced matching in complete graphs, path graphs, cycle graphs, and ladder graphs. The definition novel Line-Adjacency Matrix and its application on pro-vides foundation and a new method for future research directions in Graph Theory and Computer Science. Future research directions on the Game of Induced Matching include generalizing winning strategies and finding maximum and minimum moves for other graph types.

We also explored variations of the game where cycles, as defined in graph theory, are used in place of cycle cells, which opens the game up to non-planar graphs, such as complete graphs and gives the game a graph theory twist on top of topology.

CONCLUSION

In The Game of Induced Matching, I developed a novel method based on Line-Adjacency Matrix to predict the outcome and find all possible moves for each game. I investigated properties of induced matching on specialized graphs and patterns involving different graph types including Complete Graphs, Path Graphs, Cycle Graphs, and Ladder Graphs. I investigated and proved the playing strategies and maximum and minimum moves on these graphs. Future directions of study include finding a time-efficient alternative to the Line-Adjacency Matrix approach and investigating the relationship of game outcome related to playing orders.

Declaration by Authors

Ethical Approval: Approved

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