Galore International Journal of Applied Sciences and Humanities
Vol.1; Issue: 2; April-June 2017
Website: www.gkpublication.in/gijash.html
Research Paper *ISSN: 2456-8430*

Characterizations of Rotundities of Normed Spaces by Means of Their d - Properties

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ABSTRACT

The notion of d-property has been defined by the authors Sankar Raj V. and Eldred A. A. [5] Inspired by it, we define three other new notions called; LUR d-property, UR d - property, and URED d-property. We use these notions in the characterizations of rotund (R) spaces, locally uniformly rotund spaces, uniformly rotund spaces, and normed spaces that are uniformly rotund in every direction respectively. We use the characterizations of these rotundities to prove the best proximity point theorems of various contraction mappings.

Key words: d-properties, R norm, LUR norm, UR norm, URED norm

1. INTRODUTION

The terminology strictly convex (or rotundity) comes from the shape of the unit sphere. Historically the first result in this direction is due to J. A. Clarkson [1] who proved that every separable Banach space has an equivalent R norm. The study of strictly convex norms, should be of great interest in the near future. The concept of a rotund normed linear space has been extremely fertile. The most elementary, well-known characterizations of a strictly convex space can be found in C. R. James [3] and V. Smulian. [2] P. M. Milicic [4] characterized rotund norm by g-angle. Recently, Sanker Raj [5] has characterized rotund normed space via d-property. The concept of a locally uniformly convex norm was introduced by A.R. Lovaglia. [6] He generalized uniform convexity and localized this notion and then introduce LUR norm of a Banach space. S. Troyanski [10] and Troyanski et al, [12] stated the first characterization of existence of LUR renormings. As in now evident, the notion of LUR norm is fundamental importance for renorming theory Molto. [15] More work about LUR norm is given in M. Raja. [18] Recently, Haydon [7] has made an impressive progress on the matter showing that X is LUR renormable if X* has a dual LUR norm. The concept of uniform rotundity of the norm in a Banach space was first introduced by J. A. Clarkson [1] and has shown that for p > l the spaces L_p and l_p are uniformly convex. Uniform rotundity signalled the beginning of extensive research efforts on geometry of Banach spaces. The work on uniformly convex Banach spaces is due to G. Pisier [11] based on earlier work of R.C. James, [8] P. Enflo, [9] and P. M. Milicic [4] characterizes UR norm by g-angle. The notion of uniformly rotund in every direction (URED) norm was first used by A.L. Garkavi. [13] He proved for any closed convex set C of URED space there is at most one $x \in C$ such that

$\sup_{y \in C} ||x - y|| = \inf_{z \in C} \sup_{y \in C} ||z - y||$.

That is, for the purpose of characterizing those spaces for which every bounded set has at most one Cebysev center. It is also used by him in connection with the uniqueness of the solution of an approximation problem. In [23] it is mentioned that a Banach X is URED if and only if $\inf\{\delta(\epsilon, z) : \| z \| = 1\} > 0$ for all $\epsilon \in [0,2]$. The relation among URED, UR, LUR and R and under what conditions the space with URED norm has a fixed point is mentioned in G.R. Damai and P. M. Bajracharya. [14] It is characterized by g-angle in P. M. Milicic. [4] These rotundity properties have been studied and appeared in many literature. They play a central role in renorming theory. In this paper, we characterize these rotundities by their respective d- properties. The rotund normed space is characterized by d-property which is significantly different from that given in Sanker Raj. [5]

SOME NOTATIONS AND TERMINOLOGIES

In the sequel, (X, ||·||) is the real Banach space with norm ||·||; S(X) is the unit sphere in X; (X*, ||·||*) is the dual space of X; S(X*) is the unit sphere of X*; B(X) is the unit ball of X; B(X*) is the unit ball of X*. UR, LUR, LUR d- property, R and NCC, d(A, B) are the short forms of uniformly rotund, locally uniform rotund, locally uniform rotund d - property, rotund(strictly convex), nonempty closed convex subsets and distance between two sets A and B respectively. All undefined terms and notation are standard and can be found, for example, in Deville et al. [16]

2. SOME DEFINITIONS

Definition 2.1 Fabian et al. [21] Let *X* be a normed space. We say that a norm ||· || on a Banach space is strictly convex (or rotund) if

$$\forall x, y \in X, \| x \| = \| y \| = \| (x+y)/2 \| = 1 \Rightarrow x = y.$$

Definition 2.2 Shanker Raj. [5] Let A, B be nonempty closed convex subset of a Banach space X. Then a pair *(A, B)* is said to have the *d*-property if and only if

$$\forall x_1, x_2 \in A; y_1, y_2 \in B, \|x_1 - y_1\| = \| x_2 - y_2\| = \text{dist}(A, B) \Rightarrow \|x_1 - x_2\| = \|y_1 - y_2\|.$$

Thus, a normed linear space *X* is said to have the *d*-property if and only if every pair *(A, B)* of non- empty and closed convex subsets of *X* has the *d*-property.

Definition 2.3 Deville et al. [16] and J. Diestel. [19] A norm ||·||of X is said to be locally uniformly rotund (LUR) *at x ∈ X* if

$$\forall \{x_n\} \subseteq X, \lim_n \|x_n\| = \|x\| = \|x_n + x\| = 2\| x\| \Rightarrow \lim_n \| x_n - x \| = 0.$$

Definition 2.4 Let X be a Banach space and *A* and *B* be nonempty closed, and convex subsets of X .Then a pair (*A*, *B*) is said to have LUR d-property if

$$\|x - y\| = \lim_n \| x_n - y_n \| = \text{dist}(A, B) \Rightarrow \lim_n \| x - x_n \| = \lim_n \| y - y_n \|,$$

Whenever $x, x_n \in A$; $y, y_n \in B$ for all n . Thus a normed linear space X is said to have the LUR d-property if every pair (*A*, *B*) of subsets of X has the LUR d-property .

Definition 2.5 Fabian et al. [20] Let *X* be a normed space. We say that a norm ||·|| on a Banach space is uniformly rotund (UR) if

$$\forall \{x_n\}, \{y_n\} \subseteq X, \lim_n \|x_n\| = \lim_n \|y\| = \lim_n\|(x_n + y_n)/2\| = 1 \Rightarrow \lim_n \|x_n - y_n\| = 0.$$

Definition 2.6 A pair (A, B) of nonempty closed convex subsets of a Banach space X is said to have the UR d- property if

$$\lim_n \|x_n - u_n\| = \lim_n \| y_n - v_n \| = dist(A;B)) \in \lim_n \| x_n - y_n \| = \lim_n \| u_n - v_n \|,$$

Whenever $x_n, y_n \in A$ and $u_n, v_n \in B$ for all $n \in \mathbb{N}$. Thus a normed linear space X is said to have the UR d-property if every pair (*A*, *B*) of subsets of X has the UR d-property.

Definition 2.7 G.R. Damai and P.M. Bajracharya. [14] We say that a norm ||· || a Banach space X is said to be URED, if $x_n, y_n \in S(X); \lim_n||(x_n + y_n)/2|| = 1, x_n - y_n = z \Rightarrow z = 0.$

It is the generalization of concept of UR norm. The geometric significance of URED is the collection of all chords of the unit ball that are parallel to a fixed direction and whose lengths are bounded below by a positive number has the property that the midpoints of the chords lie uniformly deep inside the unit ball. A URED Banach space is a realm of studying minimal displacement problems.

Definition 2.8. Let X be a Banach space and A, B are nonempty closed convex subsets of X. Then we say that the pair (A, B) has URED d- property, if

$\forall\ x_n, y_n \in A;\ u_n, v_n \in B,\ 0 \neq z \in A,$
$0 \neq w \in B : x_n - y_n = z,\ u_n - v_n = w,$
$||x_n - u_n|| = ||y_n - v_n|| = d(A,B)$
$\Rightarrow ||z|| = ||w||.$

Definition.2.10 W.B. Johnson, and Lindenstrauss. [17] Renorming of Banach space consists of replacing the given norm, which is usually provided by the very definition of the space, by another norm which may have better (or sometimes worse) geometric properties of Convexity or smoothness, or both.

Remark **2.9** UR norm gives better figure than that of LUR norm and LUR gives better figure than that of R norm. So UR ⇒ LUR ⇒ R and UR ⇒URED ⇒ R but not conversely. There is no relation between LUR and URED, it is verified in G.R Damai & P.M. Bajracharya. [14]

3. Characterization of the Property (R) by d- property.

Theorem 3.1 A Banach space X is rotund if and only if X has d-property.

Proof. Let *X* be a strictly convex space. We have to show X has d-property. Let *A, B* be nonempty, closed, and convex subsets of *X*. Suppose $x_1, x_2 \in A$ and $y_1, y_2 \in B$ such that

$$\| x_1 - y_1 \| = \| x_2 - y_2 \| = \text{dist}\,(A, B).$$

We will show that $\| x_1 - x_2 \| = \| y_1 - y_2 \|$.
Put $u = x_1 - y_1$; $v = x_2 - y_2$.
Then, $||u|| = ||v|| = d(A, B)$.
So by definition of d (A, B),

$$\begin{aligned} d(A, B) &\le || (x_1+x_2)/2 - (y_1+y_2)/2 || \\ &= || (x_1-y_1)/2 + (x_2-y_2)/2|| \\ &= || u/2+v/2 || \\ &\le || u/2|| + ||v/2 || \\ &= d(A, B)/2 + d(A, B)/2 \\ &= d(A, B) \end{aligned}$$

$\therefore || (u + v)/2|| = d(A, B).$

Thus $u = v$ [∵ by rotundity of X]

$\Rightarrow x_1 - y_1 = x_2 - y_2$
$\Rightarrow x_1 - x_2 = y_1 - y_2,$
$\Rightarrow || x_1 - x_2 || = || y_1 - y_2||.$

Thus, X has d-property.

Conversely, assume that X has d-property. That is, $\forall\ x, y \in A;\ u, v \in B : ||x - u|| = || y - v|| = d(A,B) \Rightarrow ||x - y|| = ||u - v||$

We have to show X is strictly convex. Let $p, q \in X$ such that $||p|| = ||q||$. Then there are two NCC subsets *A, B in X* and $l, m \in A;\ r; s \in B$ such that

$$|| p || = || l - r || = d(A, B)$$
$$|| q || = || m - s || = d(A, B)$$
$$\begin{aligned} d(A,B) \le ||(l + m)/ 2- (r + s) / 2|| &= || (l - r)/2+(m - s)/2|| \\ &\le ||(l - r)/2|| + ||(m - s)/2|| \\ &= ||p/2|| + ||q/2|| \\ &= d(A, B)/2 + d(A, B)/2 \\ &= d(A,B). \end{aligned}$$

$\Rightarrow ||(l-r)/2+(m-s)/2|| = d(A,B))$

Thus $||l-r|| = ||m-s|| = ||(l-r+m-s)/2|| = d(A,B).$

We have to show that $l - r = m - s$.

We write,

$$0 \leq \left\| \frac{l-m}{\|l-m\|} - \frac{r-s}{\|r-s\|} \right\| \leq \left| \frac{1}{\|l-m\|} - \frac{1}{\|r-s\|} \right| (||l-m||+||r-s||)$$

∵ X has d-property we have,

$0 \leq || (l-m) - (r-s) || \leq 0$

$\Rightarrow || (l-m) - (r-s) || = 0$

$\Rightarrow (l-m) - (r-s) = 0$

$\Rightarrow l - r = m - s.$

Thus X is rotund. □

Theorem 3.2. [5] Let A, B be nonempty, closed, and convex subsets of a strictly convex Banach space X and $T: A \to B$ be a contraction mapping such that $T(A_0) \subseteq B_0$. Then there exists a unique best proximity point $x \in A$ such that

$||x - Tx|| = d(A, B).$

Further, for each fixed x_0 in A_0, there is a sequence $\{x_n\}$ such that

$\forall n \in \mathbb{N}, ||x_{n+1} - Tx_n|| = d(A, B)$ and $\{x_n\}$ converges to the best proximity point x of the map T in A.

Theorem 3.3 Let A and B be nonempty and weakly compact convex subsets of a strictly convex Banach space such that A has normal structure. Let $T: A \to B$ be a non-expansive mapping such that $T(A0) \subseteq B0$. Then T has at least one best proximity point in A, i.e., there exists $x \in A$ such that

$||x - Tx|| = d(A, B).$

4. Characterization of the Property (LUR) by LUR d- property.

Theorem 4.1. A Banach space X is locally uniformly rotund iff X has LUR d- property.

Proof. Assume that X has LUR norm. That is,

$\forall u_n = x_n - y_n, u = x - y \in X, \| u \| = \lim_n \|u_n\| = \|(u+u_n)/2\| = dist(A,B) \Rightarrow \lim_n \|(u - u_n)\| = 0.$

We have to show that X has d-property. Let $||x - y|| = \lim_n \| x_n - y_n\| = d(A, B)$. Then by definition,

$$\begin{aligned} d(A,B) \leq \lim_n \| (x + x_n)/2 - (y + y_n)/2\| &= \lim_n \|(x-y)/2 + (x_n - y_n)/2\| \\ &\leq \lim_n \|u/2\| + \lim_n \|u_n/2\| \\ &= d(A, B)/2 + d(A, B)/2 \\ &= d(A, B) \end{aligned}$$

$$\therefore \quad \lim_n \|(u + u_n)/2\| = d(A, B). \qquad (1)$$

Thus

$\|u\| = \lim_n \| u_n\| = \lim_n \|u/2 + u_n/2\| = d(A, B).$

So by LUR of X

$\Rightarrow 0 \leq \lim_n | \|x_n - y_n\| - \| x - y \| | \leq \| x_n - y_n - (x-y) \| = \| u_n - u \| = 0.$

$\Rightarrow 0 \leq \lim_n | \|x_n - y_n\| - \|x - y\| | \leq 0$

$\Rightarrow \| x - \lim_n x_n \| = \| y - \lim_n y_n \|.$

Hence X has LUR d-property.

Conversely, assume that X has the LUR d- property. That is

$\forall x, x_n \in A; y, y_n \in B, \| x - y \| = \lim_n \|x_n - y_n\| = \text{dist}(A, B) \Rightarrow \lim_n \|x_n - x\| = \lim_n \| y_n - y \|.$

We have to show that X has LUR norm.

Let p, q, p_n, q_n in X: $\|p - q\| = \lim_n \|p_n - q_n\| = \lim_n \| \frac{p-q+p_n-q_n}{2} \| = \text{dist}(A, B)$

We write,

$$0 \le \lim_n \| \frac{p_n - p}{\|p_n - p\|} - \frac{q_n - q}{\|q_n - q\|} \| \le \lim_n (| \frac{1}{\|p_n - p\|} - \frac{1}{\|q_n - q\|} |)(\|p_n - p\| + \|q_n - q\|) = 0,$$

Since *(A, B) has LUR-d* property.

So we have

$$\Rightarrow 0 \le \lim_n \|(p_n - q_n) - (p - q)\| \le 0,$$
$$\Rightarrow 0 \le \lim_n \|(p_n - q_n) - (p - q)\| = 0,$$
$$\Rightarrow \lim_n (p_n - q_n) = (p - q).$$

$\therefore \forall p, q, p_n, q_n$ in X, $\|p - q\| = \lim_n \|p_n - q_n\| = \lim_n \| \frac{p-q+p_n-q_n}{2} \| = \text{dist}(A, B$

$$\Rightarrow \lim_n (p_n - q_n) = (p - q).$$

Thus X has LUR norm. □

Remark 4.2 In [24] there was introduced p-property of a Banach space but both d-property and p-property are same only. Initially, we defined it as d-property. After that, we feel that it seems to have some parallel notion and hence we rename it as p-property.

Theorem4.3. Let *A, B, C* and *D* be nonempty subsets of a metric space (*X, d*) such that $A \subseteq C$, $B \subseteq D$ and $d(A,B) = d(C,D)$. If (*C, D*) satisfies the LUR d-property, then (*A, B*) also satisfies the LUR d-property.

Proof. Let *A, B, C* and *D* be nonempty subsets of a normed space X such that $A \subseteq C$, $B \subseteq D$ and $d(A,B) = d(C,D)$. Assume that (*C, D*) satisfies the LUR d-property. We have to prove that (A, B) satisfies the LUR d-property. Let for any x, $\{y_n\} \subseteq A$ and u, $\{v_n\} \subseteq B$ such that

$\|x - u\| = \lim_n \|y_n - v_n\| = d(A,B)$.

Since $A \subseteq C$, $B \subseteq D$ and $d(A, B) = d(C, D)$, then x; $\{y_n\} \subseteq C$ and u, $\{v_n\} \subseteq D$.

$\|x - u\| = \lim_n \|y_n - v_n\| = d(C, D)$.

By assumption, (C, D) satisfies the LUR d-property.

Therefore $\lim_n \|x - y_n\| = \lim_n \|u - v_n\|$.

Thus, (*A, B*) satisfies the LUR d-property.□

Remark 4.4 Let X has LUR d- property. Then X has d- property.

5. Characterization of the Property (UR) by UR d- property.

Theorem 5.1 A Banach space X uniformly rotund if and only if *X* has the UR d- property.

Proof. Assume that X has UR norm .That is, $\forall p_n = x_n - y_n$, $q_n = u_n - v_n \in X$,

$$\lim_n \|p_n\| = \lim_n \|q_n\| = \|(p_n + q_n)/2\| = d(A,B \Rightarrow \lim_n \|(p_n - q_n)\| = 0.$$

We have to show that X has d-property. Let $\lim_n \|x_n - u_n\| = \lim_n \|y_n - v_n\| = d(A, B)$.

$$d(A,B) \le \lim_n \| (x_n + y_n)/2 - (u_n + v_n)/2 \| = \lim_n \| (x_n - u_n)/2 + (y_n - v_n)/2 \|$$
$$\le \lim_n \| p_n/2 \| + \lim_n \| q_n/2 \|$$
$$= d(A,B)/2 + d(A,B)/2$$
$$= d(A,B)$$
$$\Rightarrow \lim_n \| (p_n + q_n)/2 \| = d(A,B).$$

Thus $\lim_n \| p_n \| = \lim_n \| q_n \| = \lim_n \| p_n/2 + q_n/2 \| = d(A,B)$.

So by LUR of X $\Rightarrow 0 \le \lim_n | \|x_n - y_n\| - \|u_n - v_n\| | \le \| x_n - y_n - (u_n - v_n) \| = \| p_n - q_n \| = 0$.

$$\Rightarrow 0 \le \lim_n | \|x_n - y_n\| - \|u_n - v_n\| | \le 0$$
$$\Rightarrow \lim_n \| x_n - y_n \| = \lim_n \| u_n - v_n \|$$

Hence X has UR d-property.

Conversely, assume that X has the LUR d- property. That is $\forall\, x_n, y_n \in A$; $u_n, v_n \in B$,

$$\|x_n - u_n\| = \lim_n \|y_n - v_n\| = \text{dist}(A,B) \Rightarrow \lim_n \| x_n - y_n \| = \lim_n \| u_n - v_n \|.$$

We have to show that X has LUR norm. Let $p_n, q_n \in X : \| p_n \| = \| q_n \|$. Then there are two nonempty closed convex subsets A, B in X and $x_n, y_n \in A$ and $u_n, v_n \in B$ such that

$$\lim_n \|p_n\| = \lim_n \|x_n - u_n\| = d(A,B)$$
$$\lim_n \|q_n\| = \lim_n \|y_n - v_n\| = d(A,B)$$

Clearly, we have $\lim_n \|(x_n - u_n + y_n - v_n)/2\| = d(A.B)$

Thus

$\forall\, x_n, y_n ; u_n, v_n$ in X, $\lim_n \|x_n - u_n\| = \lim_n \|y_n - v_n\| = \lim_n \|(x_n - u_n + y_n - v_n)/2\| = d(A,B)$.

We write,

$$0 \le \lim_n \left\| \frac{x_n - y_n}{\| x_n - y_n \|} - \frac{u_n - v_n}{\| u_n - v_n \|} \right\| \le \lim_n \left(\left| \frac{1}{\| x_n - y_n \|} - \frac{1}{\| u_n - v_n \|} \right| \right) \left(\|x_n - y_n\| + \|u_n - v_n\| \right),$$
$$= 0,$$

Since *(A, B) has LUR d* - property. So we have

$$\Rightarrow 0 \le \lim_n \| (x_n - y_n) - (u_n - v_n) \| \le 0,$$
$$\Rightarrow \lim_n \| (x_n - u_n) - (y_n - v_n) \| = 0,$$
$$\Rightarrow \lim_n (x_n - y_n) = \lim_n (u_n - v_n).$$
$$\therefore \forall u_n, v_n, x_n, y_n \text{ in X}, \lim_n \|x_n - y_n\| = \lim_n \| u_n - v_n \|$$
$$= \lim_n \left\| \frac{x_n - y_n + u_n - v_n}{2} \right. = d(A, B$$
$$\Rightarrow \lim_n (x_n - y_n) = \lim_n (u_n - v_n).$$

Thus X has LUR norm. □

Theorem 5.2. Let *A*, *B*, *C* and *D* be nonempty subsets of a metric space (*X*, *d*) such that $A \subseteq C, B \subseteq D$ and $d(A,B) = d(C,D)$. If (*C*, *D*) satisfies the LUR d-property, then (*A*, *B*) also satisfies the LUR d-property.

This theorem can be proved analogously as in Theorem 4.2.

As a consequence of UR-d property is used to prove the following theorem.

Theorem 5.3. [22] Let A and B be nonempty closed convex subsets of a uniformly convex Banach space. Suppose f : A ∪ B → A ∪ B is a cyclic contraction, that is, f(A) ⊆ B and f(B) ⊆ A and there exists k ∈ (0, 1) such that

$d(f(x), f(y)) \le k\, d(x, y)(1 - k)\, d(A,B)$ for every $x \in A$; $y \in B$.

Then there exists a unique best proximity point in A. Further, for each $x \in A$; $\{x_n\}$ converges to the best proximity point.

6. Characterization of the Property (UCED) by URED d- property.

Theorem 6.1 A Banach space X has URED norm if and only if, it has URED d-property.

Proof. Let X has URED norm. We have to show that X has URED d-property. Let A and B be any nonempty closed convex subsets of X.

Suppose that

$$x_n, y_n \in A \text{ and } u_n, v_n \in B,\ 0 \neq z \in A,\ 0 \neq w \in B : x_n - y_n = z;$$
$$u_n - v_n = w \text{ and } \|x_n - u_n\| = \lim_n \| y_n - v_n\| = d(A,B):$$

We will show that $\|z\| = \|w\|$

Put $p_n = x_n - u_n$ and $q_n = y_n - v_n$

Then, by assumption $\|p_n\| = \|q_n\| = d(A, B)$.

$$\begin{aligned} d(A,B) &\le \|(x_n + y_n)/2 - (u_n + v_n)/2\| = \|(x_n - u_n)/2 + (y_n - v_n)/2\| \\ &= \|p_n/2 + q_n/2\| \\ &\le \|p_n/2\| + \|q_n/2\| \\ &= d(A, B)/2 + d(A, B)/2 \\ &= d(A, B) \end{aligned}$$

$$\Rightarrow \lim_n \|p_n + q_n/2\| = d(A, B)$$

$\Rightarrow p_n, q_n \in X;\ \|p_n\| = \|q_n\| = \lim_n\|(p_n + q_n)/2\| = d(A,B)$,

$\lim_n (p_n - q_n) = r = (z - w) \Rightarrow r = 0$,

We write $0 \le |\,\|z\| - \|w\|\,| \le \|z - w\| = \|r\| = 0 \Rightarrow \|z\| = \|w\|$

Thus (A, B) has the URED d-property. Since $A, B \subset X$ are arbitrary nonempty closed convex subsets, X has the URED d-property.

The converse part of this theorem is proved analogously by using Theorems 5.1. □

As consequence of URED d- property we prove the following theorem;

Theorem 6.2. [14,16] Let X be a Banach space with a URED norm. Let C be a weakly compact convex subset of X and assume that T: C → C is a non expansive mapping. That is, $\|T(x) - T(y)\| \le \|x - y\|$ for every $x, y \in C$. Then there exists x_0 in C. such that $T(x_0) = x_0$.

It has been shown by V. Zizler. [25] (Proposition 14) that X can be renormed so as to be *URED* if there is a continuous one-to-one linear map T of X into a space Y that is *UCED*.Thus by Theorem 6.1, X can be renormed by URED d- property.

CONCLUSION

In this paper, the new geometrical properties called LUR d-property, UR d-property and URED d-property which characterize LUR, UR and URED normed spaces are introduced. So, these notions are very interesting and important to study because it throws light on the geometric properties of a Banach spaces. They are used in uniqueness and existence of best proximity point for various contraction mappings on the frame of these spaces. Eventually by using these d-properties we renorm many Banach spaces by them that are renormed by LUR, UR and URED norms.

ACKNOWLEDGEMENTS

I would like to express my gratitude to my Ph. D advisor, Prof. Dr. Prakash Muni Bajracharya, for his encouragement, constant support, helpful remarks and insightful comments on this paper. I also would like to express my deep gratitude to UGC of Nepal for financial support. Finally, the authors are very grateful to the referees for their valuable suggestions and opinions.

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How to cite this article: Damai GR, Bajracharya PM. Characterizations of rotundities of normed spaces by means of their d - properties. Galore International Journal of Applied Sciences & Humanities. 2016; 1(2): 21-32.

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